UNIVERSIDADE ESTADUAL DE MARINGÁ
CENTRO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA
(Doutorado)

JOÃO AUGUSTO NAVARRO COSSICH

INVARIANCE PRESSURE FOR CONTROL SYSTEMS

Maringá-PR
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Tese apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas da Universidade Estadual de Maringá, como requisito para obtenção do título de Doutor em Matemática.
Área de concentração: Geometria e Topologia.

Orientador: Prof. Dr. Alexandre José Santana.
Co-orientador: Prof. Dr. Fritz Colonius.

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Dados Internacionais de Catalogação na Publicação (CIP)
(Biblioteca Setorial BSE-DMA-UEM, Maringá, PR, Brasil)

Cossich, João Augusto Navarro
C836i Invariance pressure for control systems / João Augusto Navarro Cossich. -- Maringá, 2019.
95 f. : il.

Orientador: Profº. Drº. Alexandre José Santana.
Coorientador: Profº. Drº. Fritz Colonius.
Tese (doutorado) - Universidade Estadual de Maringá, Centro de Ciências Exatas,Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia, 2019.

1. Sistemas de controle. 2. Conjuntos controláveis. 3. Entropia de invariância. 4. Pressão de invariância. 5. Control systems. 6. Control sets. 7. Invariance entropy. 8. Invariance pressure. I. Santana, Alexandre José, orient. II. Colonius, Fritz, orient. III. Universidade Estadual de Maringá. Centro de Ciências Exatas. Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia. IV. Título.

CDD 22.ed. 515.352
Edilson Damasio CRB9-1.123

JOÃO AUGUSTO NAVARRO COSSICH

PRESSÃO INVARIANTE PARA SISTEMAS DE CONTROLE

Tese apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas da Universidade Estadual de Maringá, como parte dos requisitos necessários para a obtenção do título de Doutor em Matemática tendo a Comissão Julgadora composta pelos membros:

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Aprovada em: 13 de fevereiro de 2019.
Local de defesa: Auditório do Departamento de Matemática, Bloco F67, campus da Universidade Estadual de Maringá.

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2019

A Jesus, por toda a capacitação
para a realização deste trabalho.

Agradecimentos

Inicialmente, agradeço a Deus por toda graça e capacitação para realizar este trabalho. A Ele seja a glória, a honra e a força para sempre.

Gostaria de agradecer à minha esposa Patrícia por ser a minha melhor amiga, companheira e por sempre me encorajar ao longo do trabalho. Sua ajuda foi fundamental e crucial para a conclusão desta tese.

Eu gostaria de agradecer aos meus pais Andréa e Robson que sempre me ajudaram e investiram em minha educação. O suporte, motivação e incentivo que me deram ao longo da minha vida acadêmica são memoráveis. Agradeço também à minha avó Stella que sempre me animou e que tem sido uma inspiração desde a tenra idade.

É óbvio que também gostaria de agradecer, e muito, ao meu orientador Alexandre Santana que, desde quando ingressei na universidade, tem me orientado. Grande parte da bagagem matemática que hoje tenho aprendi com ele, além de todas as conversas e incentivos que também me motivaram e foram extremamente importantes para a realização da presente tese. Em nome dele, agradeço a todos os professores do Departamento de Matemática da UEM que, direta ou indiretamente, contribuíram para minha formação.

Sou grato também ao meu co-orientador Fritz Colonius cuja contribuição foi essencial principalmente nos problemas de maior dificuldade do trabalho, além das valorosas discussões que muito agregaram no meu conhecimento matemático.

Agradeço também aos professores Adriano da Silva, Josiney Alves, Simão Stelmastchuk e Hélio Tozatti, membros da banca examinadora, pelas contribuições para a redação final da tese.

Minha gratidão também à Lúcia Kato por seus serviços prestados a mim na Secretaria do PMA desde o início de minha graduação, sempre com muita paciência e eficiência. Agradeço também aos meus colegas de pós-graduação Ademir Pampu, Anderson Setti, Bruno Rodrigues e Richard Alves, em nome de quem agradeço aos demais colegas alunos do PMA, cujas discussões e conversas ao longo do doutorado foram de acentuado valor.

Finalmente, gostaria de agradecer à CAPES, bem como a todo o povo brasileiro, pelo apoio financeiro ao longo da minha formação.

"Sabemos que todas as coisas cooperam para o bem daqueles que amam a Deus, daqueles que são chamados segundo o seu propósito. Porquanto aos que de antemão conheceu, também os predestinou para serem conformes à imagem de seu Filho, a fim de que ele seja o primogênito entre muitos irmãos."

Romanos 8: 28-29.

Abstract

The present thesis aims to introduce the concept of invariance pressure for continuous and discrete-time control systems, a measure which generalizes the invariance entropy and can be understood as a weighted average of the total quantity of information that the controls acting on the system produces such that their trajectories starting in a subset $K \subset Q$ remain in the given set Q.

We present the main properties of this quantity, as well as lower bounds (when K and Q has positive Riemannian volume) and upper bounds (when Q is a control set). At the end of the work, we establish two variations of the concept of invariance pressure, which we call inner invariance pressure and topological feedback pressure, and we show the equivalence between these quantities for strongly invariant sets, and relate them to the transmission data rates.

Keywords: Control systems, Control sets, Invariance entropy, Invariance pressure.

Resumo

A presente tese tem por objetivo introduzir o conceito de pressão de invariância para sistemas de controle em tempo contínuo e discreto, uma medida que generaliza a entropia de invariância e pode ser entendida como um valor ponderado da quantidade total de informação que os controles que atuam sobre o sistema fornecem para que as trajetórias começando em um subconjunto $K \subset Q$ permaneça no dado conjunto Q.

Apresentaremos as principais propriedades desta quantidade, além de limitantes inferiores (quando K e Q tem volume Riemanniano positivo) e superiores (quando Q é um conjunto controlável). No fim do trabalho, estabelecemos duas variações da pressão de invariância, as quais chamamos de pressão de invariância interna e pressão topológica de feedback, e mostramos a equivalência entre estas quantias para conjuntos fortemente invariantes, além de relacioná-las com taxa de transmissão de dados.

Palavras-chave: Sistemas de controle, Conjuntos controláveis, Entropia de invariância, Pressão de invariância.

CONTENTS

INTRODUCTION

In 1865, the German physicist and mathematician Rudolf Clausius, one of the pioneers of thermodynamics, developed the concept of entropy, which is a measure of the degree of "disorder" of a thermodynamic system in equilibrium. Since then, other concepts of entropy have been proposed in several areas of science, such as in Information Theory where the American engineer Claude Shannon defines entropy as the average information associated with an alphabet $\mathcal{A}$ whose elements are symbols transmitted by communication channel.

On the one hand, along the development of ergodic theory, the Soviet mathematicians Andrey Kolmogorov and Yakov Sinai, inspired by works of Shannon, proposed in 1958-1959 (cf. Kolmogorov [29], [30] and Sinai [35]) the well-known (metric) entropy $h_\mu(T)$ of a measurable map $T : X \to X$ on a probability space X that preserves a measure of probability μ. Their objective were to provide an ergodic equivalence invariant that, for example, would allow us to distinguish two Bernoulli shifts.

As early as 1965, Adler, Konheim and McAndrew [2] introduced the topological entropy $h_{\text{top}}(T)$ of a continuous map $T : X \to X$ on a compact topological space X whose definition is given in terms of open covers of X. The mathematician Rufus Bowen provided, in 1971, a new definition of topological entropy of a continuous map T on a metric space (X, d) (not necessarily compact) via separable and spanning sets (see Bowen [4]). Such a definition allows us to interpret $h_{\text{top}}(T)$ as a precise numerical measure of the global exponential complexity in the orbital structure of a topological dynamic system.

In order to establish a concrete relationship between $h_\mu(T)$ and $h_{\text{top}}(T)$, in the years 1970 and 1971, Dinaburg [18], [19] and Goodman [20] have proved the variational principle for continuous maps T over a compact metric space X:

$$h_{\text{top}}(T) = \sup\{h_\mu(T)\},$$

where the supremum is taken over all measures of probability μ invariant by T.

On the other hand, the centenary theory of Equilibrium Statistical Mechanics was consolidating. Briefly, this theory aims to understand, through a probabilistic approach, the relations between the salient macroscopic characteristics observed in a system of particles at equilibrium and the properties of their microscopic constituents. We can visualize this in the Ising model which, in the magnetization study, is described by a lattice that is a finite subset Λ of $\mathbb{Z}^d$ where at each point x of the lattice we can associating the values ± 1 (-1 means that the particle x has spin down, $+1$ means that the spin points up).

Thus, a system configuration is an element of the configuration space $\Omega := \{-1, +1\}^\Lambda$, and an equilibrium state, mathematically represented by a measure of probability in Ω, describes a macroscopic configuration of the system that can be physically observed, such as Gibbs free energy, heat or pressure, many of which can be viewed as weighted averages of the quantities defined in terms of the constituents of the system (the sum of the energies of the molecules, for example). In addition, the equilibrium states are characterized by minimizing these macroscopic configurations.

Since ergodic theorems depend on a previously fixed invariant measure by the system, one of the problems of Equilibrium Statistical Mechanics is to choose an invariant measure to analyze the system in order to apply the results of ergodic theory. Another relevant problem in this area is that: as macroscopic configurations of the system, such as pressure, can be seen mathematically as weighted averages, what weight over the constituents of the system should we use?

To exemplify what we are saying, in the Ising model, suppose that the set Ω is finite and write $\Omega = \{\omega_1, \ldots, \omega_n\}$. Let $f(\omega_i)$ be the total energy of the system in the state ω_i. In this case, f would be a weight assigned to the configuration ω_i of the molecules of the system and

$$\mu(\omega_i) := \frac{e^{-\beta f(\omega_i)}}{Z(\beta)}$$

is called the Gibbs measure, which minimizes the free energy of the system in question, where $\beta = (k_B T_s)^{-1}$ with k_B being the Boltzman constant, T_s the temperature of the system in equilibrium and $Z(\beta) = \sum_{i=1}^{n} e^{-\beta f(\omega_i)}$.

An excellent tool for the problem above (where weights and probability measures are considering) was developed by the Belgian physicist and mathematician David Ruelle who introduced in [34] the concept of pressure for expansive actions on compact subsets of $\mathbb{Z}^d$ and proved the variational principle for the pressure, thus providing a good way of choosing measures that minimize macroscopic configurations of particles systems, which are precisely the states of equilibrium. Then, this concept was generalized by Peter Walters in [38] for continuous maps on compact metric spaces, formalizing what we know today as topological pressure of a continuous maps. Since then, this idea of pressure has been adapted in other contexts, such as Pesin and Pitskel [33], Bogenschütz [3], Zeng, Yan and Zhang [22] and [41].

Already in the control systems environment, the seminal article of Nair, Evans, Mareels and Moran [32] gave rise to the concept of topological feedback entropy as an inherent measure for the rate at which a control system generates stability information. This concept was defined for discrete-time control systems in terms of invariant covers of a given subset K of the state space X, based on ideas similar to those presented in [2] for continuous maps on compact spaces.

Inspired by the concept of topological feedback entropy, in 2009 Colonius and Kawan [9] introduced the invariance entropy of a admissible pair (K, Q) for continuous-time control systems in terms of spanning sets, similarly to [4]. This quantity measures the exponential growth rate of the minimal number of different control functions sufficient for that the trajectories of the system to remain in Q when they start in $K \subset Q$, as time tends to infinity. The relation between these two types of entropy was established in [10] by Colonius, Kawan and Nair, where it is shown that these two quantities coincide in the case of strongly invariant sets. The invariance entropy has been widely studied since then, as we can see in Kawan [25], da Silva [13], [14], Colonius, Fukuoka & Santana [8] and da Silva & Kawan [15], [16].

In this work we introduce the invariance pressure for control systems in both discrete and continuous-time. In order to give an overall outline of invariance pressure, consider the continuous-time control system

$$\dot{x}(t) = F(x(t), \omega(t)),\ \omega \in \mathcal{U}$$

on a differentiable manifold M. Denoting by $U \subset \mathbb{R}^m$ the set where the controls $\omega : \mathbb{R} \to U$ take values, we can consider a weight $f : U \to \mathbb{R}$ on these control values in the following way: given $\omega \in \mathcal{U}$ and $\tau > 0$, let $(S_\tau f)(\omega) := \int_0^\tau f(\omega(t))dt$. Thus, if (K, Q) is an admissible pair and $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)-spanning set, then $\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)}$ represents the total quantity of weighted information that the controls in $\mathcal{S}$ produce up to time τ so that the system remains in Q when it starts in K. This information depends obviously on the weight f and what the control functions represents in the system.

In order to optimize this quantity, that is, the weighted information necessary for accomplishing the control task on $[0, \tau]$, consider

$$a_\tau(f, K, Q) := \inf\left\{\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S}\ (\tau, K, Q)\text{-spanning set}\right\}.$$

The invariance pressure of the system is defined as

$$P_{\text{inv}}(f, K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q)$$

and it can be interpreted as the exponential growth rate of the quantity of total weighted information produced by the control functions acting on the system in order to its trajectories remains in Q, once started in K, as time tends to infinity. When we do not weight the control values, that is, if $f \equiv \mathbf{0}$, where $\mathbf{0}$ is the null function, then $P_{\text{inv}}(\mathbf{0}, K, Q)$ coincides with the (strict) invariance entropy defined in [9], hence the invariance pressure is a generalization of the concept of invariance entropy.

This work is divided as follows: In Chapter 1 we have established the basic notions for the development of the thesis, such as differentiable manifolds, control systems and invariance entropy. In this sense, both continuous and discrete-time control systems are presented, as well as the invariance entropy for each of these systems with their respective variations, such as the topological feedback entropy and the outer and inner invariance entropy.

In Chapter 2 we introduce the main concept of this work: the invariance pressure. In this chapter we will study the pressure only for continuous-time control systems and explore the basic and elementary properties which this quantity has in relation to each of its arguments, making a parallel with the properties of the topological pressure for dynamic systems, in addition to proving the invariance of this amount by time-invariant conjugacy.

Also for continuous-time, in Chapter 3 we derive some results on the computation of invariance pressure $P_{\text{inv}}(f, K, Q)$ of a admissible pair (K, Q) and $f : U \to \mathbb{R}$, where Q satisfies some particular control properties, such as isolated sets, inner control sets and control sets. The last section of this chapter is devoted to presenting some lower bounds for $P_{\text{inv}}(f, K, Q)$ for systems over a Riemannian manifold, where both K and Q have positive Riemannian volume.

In Chapter 4 of the present thesis aims to prove the Theorem 4.2.3 that generalizes the Theorem 3.1 of [10] that relates the topological feedback entropy and the invariance entropy of strongly invariant sets. For this, we define, in the first two sections, the concepts of topological feedback pressure and the inner invariance pressure in order to generalize those concepts of entropy, respectively. The last section of this chapter generalizes the concept of data rate presented in [32] in order to establish a relationship between data rates and topological feedback pressure.

CHAPTER 1

PRELIMINARIES

This chapter is dedicated to establish some notions on discrete and continuous-time control systems and invariance entropy, which are the main objects of this work. Since the space state of a continuous-time control system is a smooth manifold or, in some cases, a Riemannian manifold, in the first section we provide the necessary background on these objects which is needed in this work and in the second section we present a short comment about topological pressure witch generalizes the topological entropy (for dynamical systems) as well as the invariance pressure generalizes the invariance entropy for control systems.

1.1 Differentiable Manifolds

The main goal of this section is to establish the notations and the elementary notions of differentiable manifolds necessary for the present work. The main references used in this section are Abraham et al. [1], Lima [31] and do Carmo [17].

Let M be a topological space. A **chart** for M is a pair (ϕ, U) such that U is an open subset of M and the map $\phi : U \to V$ is a homeomorphism onto an open subset V of $\mathbb{R}^d$ for some $d \in \mathbb{N}$, which is called the **dimension** of (ϕ, U). A chart (ϕ, U) is said to be a **chart around** $x \in M$ if $x \in U$. A family of d-dimensional cards $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in \Lambda}$ on M is called a C^∞**-atlas** of dimension d if

i) $\{U_\alpha\}_{\alpha\in\Lambda}$ covers of M;

ii) For all $\alpha, \beta \in \Lambda$, the transition map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is of class C^∞.

A d-dimensional chart (ϕ, U) for M is **admissible** relatively to an atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ of dimension d on M if for all $\alpha \in \Lambda$ with $U \cap U_\alpha \neq \emptyset$, the transition maps $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are of class C^∞. We say that a d-dimensional atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ on M is **maximal** if all admissible d-dimensional charts relatively to $\mathcal{A}$ are contained in $\mathcal{A}$.

A pair $(M, \mathcal{A})$ is a d-dimensional **differentiable manifold** (of class C^∞) or simply **manifold** if the set M a second-countable Hausdorff space provided with a maximal C^∞-atlas of dimension d. The natural number d is called the **dimension** of M and $\mathcal{A}$ is a **differentiable structure** on M.

It is easy to see that the d-dimensional euclidean space $\mathbb{R}^d$ is a manifold. Every open subset N of a d-dimensional manifold $(M, \mathcal{A})$ is itself a d-dimensional manifold with atlas $\{(\phi|_{U\cap N}, U \cap N);\ (\phi, U) \in \mathcal{A}\}$. Given two differentiable manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ of dimensions k and l , respectively, their Cartesian product $M \times N$ (endowed with the product topology) becomes a $(k + l)$-dimensional manifold with the maximal atlas which contains the product atlas $\{(\phi \circ \psi, U \times V);\ (\phi, U) \in \mathcal{A} \text{ and } (\psi, V) \in \mathcal{B}\}$. A manifold of this type is called a **product manifold**.

A map $f : M \to N$ between two differentiable manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ is **differentiable** at $x \in M$ if there are charts $(\phi, U) \in \mathcal{A}$ around x and $(\psi, V) \in \mathcal{B}$ around $f(x)$ such that $f(U) \subset V$ and the **local representation** $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ of f is differentiable (of class C^∞) on $\phi(x)$. We can see that this definition does not depend on the choice of charts. We say that $f : M \to N$ is **differentiable** if it is at all $x \in M$. A differentiable bijection $f : M \to N$ whose inverse $f^{-1} : N \to M$ is also differentiable is called **diffeomorphism**.

A **curve** on a manifold M is a differentiable map $c : I \to M$ defined on an open interval I of $\mathbb{R}$. Let x be an element of a d-dimensional manifold $(M, \mathcal{A})$ and denote by $\mathcal{C}_x$ the set of all curves $c : I \to M$ such that $0 \in I$ and $c(0) = x$. Given $c \in \mathcal{C}_x$ and a chart $(\phi, U) \in \mathcal{A}$ around x, whenever we write $\phi \circ c$ we are admitting that the domain of c was consciously reduced to a smaller open interval I', containing 0, such that $c(I') \subset U$. On $\mathcal{C}_x$ we can define the following equivalence relation $\sim$:

$$c_1 \sim c_2 \;\Leftrightarrow\; \exists\, (\phi, U) \in \mathcal{A}\,;\; \frac{d}{dt}(\phi \circ c_1)(t)\Big|_{t=0} = \frac{d}{dt}(\phi \circ c_2)(t)\Big|_{t=0}.$$

It is worth noting that the equality $\frac{d}{dt}(\phi \circ c_1)(t)|_{t=0} = \frac{d}{dt}(\phi \circ c_2)(t)|_{t=0}$ is independent on the choice of chart. Then we say that the equivalence class $[c]$ of a curve $c \in \mathcal{C}_x$ is a **tangent vector at** x. The quotient set $\mathcal{C}_x/\sim$ is indicated by T_xM and is called **tangent space at** x.

The set T_xM has a natural structure of a real vector space by requiring that given $(\phi, U) \in \mathcal{A}$, the well-defined bijection $\bar{\phi} : T_xM \to \mathbb{R}^d$, $\bar{\phi}([c]) = \frac{d}{dt}(\phi \circ c)(t)|_{t=0}$ is an isomorphism. The operations so defined does not depend on the choice of (ϕ, U). The preimages of the standard basis vectors $e_1, \ldots, e^d \in \mathbb{R}^d$ under $\bar{\phi}$ form a basis of T_xM. They are denoted by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$.

The set

$$TM := \bigcup_{x \in M} (\{x\} \times T_xM)$$

is the **tangent bundle** of M and it can be endowed with an atlas in a canonical way such that it becomes a $2d$-dimensional manifold (see [1, Section 3.3]).

Given a differentiable map $f : M \to N$ between the manifolds M and N, the **derivative of** f **at** $x \in M$ is the well-defined linear map $\mathrm{d}_x f : T_xM \to T_{f(x)}N$ given by $\mathrm{d}_x f([c]) = [f \circ c]$.

A **vector field** on a manifold M is a map $X : M \to TM$ such that for each $x \in M$, $X(x) \in T_xM$. The vector field X on M is differentiable (of class C^∞) if $X : M \to TM$ is differentiable as a map between the manifolds M and TM. This concept allows us to study ordinary differential equations on a manifold: given a vector field X on M, there is an ordinary differential equation associated to X in the natural way

$$\frac{d}{dt}x(t) = X(x(t)).$$

A **Riemannian metric** on a connected manifold M is a correspondence g which associates to each point $x \in M$ an inner product $g(\cdot,\cdot)_x$, that is, a symmetric, bilinear, positive-definite form on the tangent space T_xM which varies differentiably in the following sense: if (ϕ, U) is a chart around x with $y = \phi^{-1}(x_1, \ldots, x_d)$ and

$d_y\phi^{-1}(0,\ldots,1,\ldots,0) = \frac{\partial}{\partial x_i}(y)$, then $g_{i,j}(x_1,\ldots,x_d) := g(\frac{\partial}{\partial x_i}(y), \frac{\partial}{\partial x_j}(y))_y$ is a differentiable function on $\phi(U)$. It is not difficult to see that this definition does not depend on the choice of chart. It is possible to show that all differentiable manifold has a Riemannian metric. A **Riemannian manifold** is a manifold M endowed with a Riemannian metric g, and we represent it by a pair (M,g).

Let (M,g) be a d-dimensional Riemannian manifold. To each chart (ϕ,U) of M one can associate d^3 differentiable functions $\Gamma^k_{i,j} : U \to \mathbb{R}$ by

$$\Gamma^k_{i,j} := \frac{g_{k,l}}{2}\left(\frac{\partial g_{i,l}}{\partial \phi^j} + \frac{\partial g_{j,l}}{\partial \phi^i} + \frac{\partial g_{i,j}}{\partial \phi^l}\right).$$

These functions are called the **Christoffel symbols** of (M,g) with respect to the chart (ϕ,U) and they satisfies $\Gamma^k_{i,j} = \Gamma^k_{j,i}$. Using the Christoffel symbols, one can define the **Levi–Civita connection** associated with (M,g), which is an operator assigning to a pair (X,Y) of vector fields a vector field $\nabla_X Y$. Locally, we can write any vector fields X and Y as $X = \sum_{i=1}^{d} X^i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{d} Y^j \frac{\partial}{\partial x_j}$. Then $\nabla_X Y$ is defined by

$$(\nabla_X Y)(x) = X^i(x)\left(Y^j(x)\Gamma^k_{i,j}(x)\frac{\partial}{\partial x_k}(x) + \frac{\partial Y^j}{\partial x_i}(x)\frac{\partial}{\partial x_j}(x)\right).$$

Hence, given a vector field X on M, we can assign its **covariant derivative** at $x \in M$, $\nabla X(x) : T_xM \to T_xM$ by $\nabla X(x)v := (\nabla_v X)(x)$. Moreover, one can define the **divergence** of a vector field X on M by $\mathrm{div}X(x) := \mathrm{tr}\,(\nabla X(x))$ which is a differentiable function from M to $\mathbb{R}$.

The **Riemannian volume** of a Borel set A of a d-dimensional Riemannian manifold (M,g) which is contained in the domain of a chart (ϕ,U) is defined as

$$\mathrm{vol}(A) := \int_{\phi(A)} \sqrt{\det\left[g_{i,j}(\phi^{-1}(x))\right]}dx,$$

where the integral is the usual Lebesgue integral on $\mathbb{R}^d$. This definition is independent of the chosen chart. Then vol is extended naturally to all Borel subsets of M and it holds that

$$\int_{f(A)} \varphi \, d\mathrm{vol} = \int_A \varphi \circ f|\det \mathrm{d}f| d\mathrm{vol},$$

for all diffeomorphism $f : M \to M$ and all integrable function $\varphi : M \to \mathbb{R}$.

Remark 1.1.1. *For a real nonsingular $d \times d$ matrix A, $|det\ A|$ is given in terms of the product between the singular values of A (see [12, Section 11.3]).*

1.2 Topological Pressure

In this section we present the concept of topological pressure of a continuous map $T : X \to X$, where (X, d) is a compact metric space. The main references that we use here are Walters [39] and Viana and Oliveira [37].

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that a subset F of X is a (n, ε)**-spanning set** for X with respect to T if

$$\forall\, x \in X,\ \exists\, y \in F\ ;\ d(T^i x, T^i y) < \varepsilon,\ \forall\, i \in \{0, \dots, n-1\}.$$

Denote by $C(X, \mathbb{R})$ the space of real-valued continuous function of X. For $f \in C(X, \mathbb{R})$ and $n \in \mathbb{N}$, we denote $\sum_{i=0}^{n-1} f(T^i x)$ by $(S_n f)(x)$. Then, if $f \in C(X, \mathbb{R})$, $n \in \mathbb{N}$ and $\varepsilon > 0$ put

$$Q_n(T, f, \varepsilon) := \inf \left\{ \sum_{x \in F} e^{(S_n f)(x)};\ F \text{ is a } (n, \varepsilon)\text{-spanning set} \right\}.$$

Now, define

$$Q(T, f, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log Q_n(T, f, \varepsilon).$$

Definition 1.2.1. *The* **topological pressure of T relative to f**, *denoted by $P_{\text{top}}(T, f)$, is defined as*

$$P_{\text{top}}(T, f) := \lim_{\varepsilon \to 0} Q(T, f, \varepsilon).$$

Note that if $\mathbf{0}$ is the null function in $C(X, \mathbb{R})$, then $Q_n(T, \mathbf{0}, \varepsilon)$ coincides with the smallest cardinality of any (n, ε)-spanning set for X with respect to T, which we denote by $r_n(\varepsilon, X)$. Hence the topological pressure of T relative to $\mathbf{0}$, $P_{\text{top}}(T, \mathbf{0})$, is equal to the topological entropy of T, which we denote by $h_{\text{top}}(T)$ (see [39, Sect. 7.2]).

We also can get the topological pressure of T in another way: given a natural number n and $\varepsilon > 0$ we say that $E \subset X$ is a (n, ε)**-separated set** of X with respect to T if

$$\forall\, x, y \in E,\ x \neq y \Rightarrow d(T^i x, T^i y) \geq \varepsilon,\ \forall\, i \in \{0, \dots, n-1\}.$$

For $f \in C(X, \mathbb{R})$, $n \in \mathbb{N}$ and $\varepsilon > 0$ put

$$P_n(T, f, \varepsilon) := \sup \left\{ \sum_{x \in E} e^{(S_n f)(x)};\ E \text{ is a } (n, \varepsilon)\text{-separated set} \right\}.$$

Note that if $\mathbf{0} \in C(X, \mathbb{R})$ is the null function on X and $s_n(\varepsilon, X)$ denote the largest cardinality of any (n, ε)-separated subset of X with respect to T, then $P_n(T, \mathbf{0}, \varepsilon) = s_n(\varepsilon, X)$. Now, let

$$P(T, f, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, f, \varepsilon).$$

The equivalence between the definitions of topological pressure via spanning and separated sets is given in the following theorem whose proof can be found in [39, Theorem 9.1].

Theorem 1.2.2. *If $f \in C(X, \mathbb{R})$, then $P_{\text{top}}(T, f) = \lim_{\varepsilon \to 0} P(T, f, \varepsilon)$.*

1.3 Control Systems

This section is based on the references of Colonius and Kliemann [11], Sontag [36] and Kawan [27]. Here, we present the basic notions of continuous and discrete-time control systems for our work.

1.3.1 Continuous-time control systems

A **continuous-time control system** on a connected smooth manifold M is given by a family of differential equations

$$\dot{x}(t) = F(x(t), \omega(t)),\ \omega \in \mathcal{U} \tag{1.3-1}$$

parametrized by **control functions** (or simply **controls**) $\omega : \mathbb{R} \to \mathbb{R}^m$ that lies in a **set of admissible control functions**

$$\mathcal{U} := \{\omega : \mathbb{R} \to \mathbb{R}^m;\ \omega(t) \in U a.e.\} \tag{1.3-2}$$

where U is a nonempty **compact** subset of $\mathbb{R}^m$ called **control-valued space**. We require that the map $F : M \times \mathbb{R}^m \to TM$ is differentiable and, for each $u \in U$, the map

$F_u(\cdot) := F(\cdot, u)$ is a differentiable vector field on M. In the literature, it is usual to call the map F the **right-hand side** of the control system.

Once we fix an initial condition $x_0 \in M$ and a control $\omega \in \mathcal{U}$, the assumptions on F implies that there is a (locally) unique solution $\varphi(\cdot, x, \omega)$ of 1.3-1 with $\varphi(0, x, \omega) = x_0$. Since in the study of invariance entropy and invariance pressure are considered only solutions that does not leave a compact set (or a ε-neighborhood of it), we may assume that all the solutions are defined for all $t \in \mathbb{R}$, allowing us to define the map

$$\begin{array}{rccc} \varphi : & \mathbb{R} \times M \times \mathcal{U} & \longrightarrow & M \\ & (t, x, \omega) & \longmapsto & \varphi(t, x, \omega) \end{array}. \tag{1.3-3}$$

Along the text we denote by $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ a continuous-time control system 1.3-1.

In many cases we will fix one or two of the arguments of φ. In order to stress which arguments are fixed, we use the notation $\varphi_\omega(t, x) = \varphi_{t,\omega}(x) = \varphi_t(x, \omega) = \varphi(t, x, \omega)$. For each $\omega \in \mathcal{U}$, the map $\varphi_\omega : \mathbb{R} \times M \to M$ is continuous and if $t \in \mathbb{R}$, then $\varphi_{t,\omega} : M \to M$ is a homeomorphism (see [24, Theorem 1.2.10 and Corollary 1.2.12]). Moreover, for $t > 0$, $\varphi(t, x, \omega)$ does not depend on the values of ω outside of $[0, t)$.

Example 1.3.1. *Denote by $\mathbb{R}^{m_1 \times m_2}$ the set of all $m_1 \times m_2$ matrix with entries in $\mathbb{R}$. A control system in $M = \mathbb{R}^d$ is **linear** if the dynamics 1.3-1 is given by*

$$\dot{x}(t) = Ax(t) + B\omega(t), \tag{1.3-4}$$

where $A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{d \times m}$. For $x \in \mathbb{R}^d$ and $\omega \in \mathcal{U}$, we can see that, in this case, the (unique) solution of 1.3-4 such that $\varphi(0, x, \omega) = x$ is

$$\varphi(t, x, \omega) = e^{tA} + \int_0^t e^{(t-s)A} B\omega(s) ds.$$

Example 1.3.2. *Let $X_0, X_1, \cdots, X_m$ be differentiable vector fields on M. A control system 1.3-1 is called **affine** if the control range U is a compact and convex set and the right-hand side F has the form*

$$F(x, u) = X_0(x) + \sum_{i=1}^{m} u_i X_i(x),$$

*where $u = (u_1, \cdots, u_m) \in U$. The vector field X_0 is called the **drift vector field** and*

$X_1, \cdots, X_m$ *the control vector fields of the control system, respectively.*

We can define a continuous-time dynamical system θ on the set $\mathcal{U}$ putting $\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$, $\theta(t,\omega) := \theta_t\omega$, where $\theta_t\omega(\cdot) := \omega(\cdot + t)$. The map θ is usually called **shift flow**.

Remark 1.3.3. *In the affine control system case, if we assume that U is convex and $\mathcal{U}$ is a subset of the set of all the essentially bounded functions $L^\infty(\mathbb{R}, \mathbb{R}^m)$ (which is the dual of the set of the integrable functions $L^1(\mathbb{R}, \mathbb{R}^m)$), then $\mathcal{U}$ is compact and metrizable in the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^*$. A metric compatible with the topology is given by*

$$d_{\mathcal{U}}(\omega, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\left| \int_{\mathbb{R}} \langle \omega(t) - \nu(t), x_k(t) \rangle dt \right|}{1 + \left| \int_{\mathbb{R}} \langle \omega(t) - \nu(t), x_k(t) \rangle dt \right|},$$

where $\{x_k\}$ is an arbitrary countable dense subset of $L^1(\mathbb{R}, \mathbb{R}^m)$ and $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathbb{R}^m$. Moreover, the shift flow θ is continuous (see [11, Lemma 4.2.1 and Lemma 4.2.4]).

Remark 1.3.4. *The map φ satisfies the cocycle property, that is, for each $s, t \in \mathbb{R}$, $x \in M$ and $\omega \in \mathcal{U}$*

$$\varphi(t+s, x, \omega) = \varphi(t, \varphi(s, x, \omega), \theta_s\omega).$$

Remark 1.3.5. *In the case of affine control systems, we can define the* **control flow** $\Phi : \mathbb{R} \times M \times \mathcal{U} \to M \times \mathcal{U}$ *of Σ, which is given by the skew product of the solution $\varphi(t, x, \omega)$ and the shift flow, i.e.,*

$$\Phi(t, (x, \omega)) = (\varphi(t, x, \omega), \theta_t\omega).$$

In fact, under some assumptions, Φ defines a continuous dynamical system on $M \times \mathcal{U}$ (see [11, Lemma 4.3.2]).

From now, we introduce some useful qualitative notions in order to analyze the behavior of a control system. Given $\tau > 0$ and $x \in M$, we start by defining the **set of points reachable from x up to time τ**:

$$\mathcal{O}^+_{\leq\tau}(x) := \{y \in M;\ \exists\, t \in [0, \tau],\ \omega \in \mathcal{U} \text{ with } y = \varphi(t, x, \omega)\}.$$

The set $\mathcal{O}^+(x) := \bigcup_{\tau>0} \mathcal{O}^+_{\leq\tau}(x)$ is called the **positive orbit of x**. Moreover, we define the **set of points controllable to x within time τ** and the **negative orbit of x**, respectively, by

$$\mathcal{O}^-_{\leq\tau}(x) := \{y \in M;\ \exists\, t \in [0,\tau],\ \omega \in \mathcal{U} \text{ with } x = \varphi(t,y,\omega)\},\ \text{and } \mathcal{O}^-(x) := \bigcup_{\tau>0} \mathcal{O}^-_{\leq\tau}(x).$$

In studying the controllability of a control system Σ, the notion of local accessibility plays a key role, mainly to explore some properties of control sets (which we will introduce below). We say that Σ is **locally accessible from** $x \in M$ if $\text{int}\mathcal{O}^{\pm}_{\leq\tau}(x)$ are both nonempty, for all $\tau > 0$. If this condition holds for all $x \in M$, then we say that Σ is **locally accessible**.

A set $Q \subset M$, is called **controlled invariant** if for all $x \in Q$ there exists $\omega \in \mathcal{U}$ such that $\varphi(t,x,\omega) \in Q$ for all $t \geq 0$.

Remark 1.3.6. *For an affine control system Σ and a compact controlled invariant set $Q \subset M$, the forward lift of Q to $M \times \mathcal{U}$ defined as*

$$\mathcal{Q} := \{(x,\omega) \in M \times \mathcal{U};\ \varphi(\mathbb{R}_+, x, \omega) \subset Q\},$$

is a compact forward-invariant set for the control flow, that is, $\Phi_t(\mathcal{Q}) \subset \mathcal{Q}$ for all $t \geq 0$ (see [27, Proposition 1.10] for the proof).

A controlled invariant set $D \subset M$ is called a **control set** for Σ if satisfies $D \subset \overline{\mathcal{O}^+(x)}$ for all $x \in D$ (approximate controllability) and D is a maximal controlled invariant set with this property. The following result, whose proof can be found in [11, Proposition 3.2.3 and Lemma 3.2.13], collect some good properties of control sets.

Lemma 1.3.7. *For a control set $D \subset M$, the following holds:*

i) D satisfies the **no-return property**, *i.e.,*

$$\forall x \in D\, \forall \tau > 0\, \forall \omega \in \mathcal{U} : \varphi(\tau,x,\omega) \in D \Rightarrow \varphi([0,\tau],x,\omega) \subset D.$$

If additionally D has nonempty interior, then it holds that

ii) If Σ is locally accessible on M, then D is connected and $\overline{\text{int}D} = \overline{D}$.

iii) If Σ is locally accessible from $y \in \text{int}D$, then $y \in \mathcal{O}^+(x)$ for all $x \in D$.

iv) If Σ is locally accessible from all $y \in \text{int}D$, then $\text{int}D \subset \mathcal{O}^+(x)$ for all $x \in D$, and for every $y \in \text{int}D$ one has $D = \overline{\mathcal{O}^+(x)} \cap \mathcal{O}^-(x)$.

A sufficient condition to ensure local accessibility of a control system is that it satisfies the **accessibility rank condition**: set $\mathcal{F} = \{F_u;\ u \in U\}$ and denote by $\mathcal{L}_\mathcal{F}$ the Lie algebra generated by $\mathcal{F}$, the accessibility rank condition requires that

$$\mathcal{L}_\mathcal{F}(x) = T_x M,\ \ \forall\, x \in M,$$

where $\mathcal{L}_\mathcal{F}(x) = \text{span}\{F(x);\ F \in \mathcal{L}_\mathcal{F}\} \subset T_x M$. This result is known as Krener's Theorem and is stated as:

Theorem 1.3.8. *If the accessibility rank condition holds for a control system 1.3-1, then it is locally accessible.*

Example 1.3.9. *In the linear case described in Example 1.3.1, the accessibility rank condition is equivalent to the rank of the Kalman's matrix $[B\ AB\ \cdots\ A^{d-1}B]$ to be equal to d. If this happens, we say that the matrix pair (A, B) is* **controllable**.

From Hinrichsen and Pritchard [21, Theorems 6.2.19 and 6.2.20] (cf. also Colonius and Kliemann [11, Example 3.2.16]) we get the following result on existence and uniqueness of a control set for a linear control system.

Theorem 1.3.10. *Consider the linear control system described in the Example 1.3.1. Assume that the pair (A, B) is controllable and the control range U is a compact neighborhood of the origin.*

(i) Then there is a unique control set D with nonempty interior, it is convex and satisfies

$$0 \in \text{int}D \text{ and } D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)} \text{ for every } x \in \text{int}D.$$

(ii) D is closed if and only if $\mathcal{O}^+(x) \subset D$ for all $x \in D$.

(iii) The control set D is bounded if and only if A is hyperbolic, that is, all the eigenvalues of A have non zero real parts.

1.3.2 Discrete-time control systems

This subsection presents the definition of discrete-time control systems. Here, the main references are [27], [36] and [32].

Consider a metric space (X, ϱ), a topological space U and a map $F : X \times U \to X$ such that for each $u \in U$ the map $F_u(\cdot) := F(\cdot, u)$ is continuous. Define $\mathcal{U} := U^{\mathbb{N}_0}$ as the set of all sequences $\omega = (u_k)_{k\in\mathbb{N}_0}$ of elements in the control range U. A **discrete-time control system** is described by the difference equation

$$x_{k+1} = F(x_k, u_k), k \in \mathbb{N}_0 = \{0, 1, \ldots\}, \tag{1.3-5}$$

We endow $\mathcal{U}$ which is the set of control sequences with the product topology. Sometimes, we will assume that the set of control values U is a compact metric space, implying that also $\mathcal{U}$ is a compact metrizable space. The shift θ on $\mathcal{U}$ is defined by $(\theta\omega)_k = u_{k+1}, k \in \mathbb{N}_0$. For $x_0 \in X$ and $\omega \in \mathcal{U}$ the corresponding solution of (1.3-5) will be denoted by

$$x_k = \varphi(k, x_0, \omega), k \in \mathbb{N}_0,$$

and $\varphi(k, x_0, \omega)$ can be expressed as

$$\varphi(k, x, \omega) = \begin{cases} x, & \text{if} \quad k = 0 \\ F_{u_{k-1}} \circ \cdots \circ F_{u_1} \circ F_{u_0}(x), & \text{if} \quad k \geq 1. \end{cases}$$

We usually denote by $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$ a discrete-time control system 1.3-5. Conveniently, we write $\varphi_{k,\omega}(\cdot) := \varphi(k, \cdot, \omega)$. By induction, one sees that this map is continuous. Observe that this is a cocycle associated with the dynamical system on $\mathcal{U} \times X$ given by

$$\Phi(k, \omega, x_0) = (\theta^k\omega, \varphi(k, x_0, \omega)), k \in \mathbb{N}_0, \omega \in \mathcal{U}, x_0 \in X.$$

We note the following property which is of independent interest (it is not used in the rest of the thesis).

Proposition 1.3.11. *The shift θ is continuous and, if $F : X \times U \to X$ is continuous, then Φ is a continuous dynamical system.*

Proof. Continuity of θ follows since the sets of the form

$$W = W_0 \times W_1 \times \cdots \times W_N \times U \times \cdots \subset U^{\mathbb{N}_0}$$

with $W_i \subset U$ open for all i and $N \in \mathbb{N}$ form a subbasis of the product topology and the preimages

$$\theta^{-1}W = U \times W_0 \times W_1 \times \cdots \times W_N \times U \times \cdots$$

are open. If F is continuous, then induction shows that $\varphi(k, x_0, \omega)$ is continuous in $(x_0, \omega) \in X \times \mathcal{U}$ for all k. $\square$

1.4 Invariance Entropy

In this section we introduce initially the invariance entropy for continuous-time control systems and present some results and properties which can be found in [27]. Next, the concept of inner invariance entropy and topological feedback entropy are presented for discrete-time control systems, and the relation of these two concepts. We start with the definition of admissible pair which will be used along this work.

Definition 1.4.1. *A pair (K, Q) of nonempty subsets of M is called* **admissible** *for the control system $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ if it satisfies the following properties:*

i) K is compact;

ii) For each $x \in K$, there exists $\omega \in \mathcal{U}$ such that $\varphi(t, x, \omega) \in Q$ for all $t \geq 0$.

Given $\tau > 0$ and an admissible pair (K, Q), we say that a set $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)**-spanning set** if

$$\forall\, x \in K,\ \exists\, \omega \in \mathcal{S};\ \varphi([0, \tau], x, \omega) \subset Q.$$

Denote by $r_{\text{inv}}(\tau, K, Q)$ the minimal number of elements such a set can have (if there is no finite set we say that $r_{\text{inv}}(\tau, K, Q) = \infty$). If $K = Q$ we omit the argument K, that is, we write $r_{\text{inv}}(\tau, Q)$ and speak (τ, Q)-spanning set.

The existence of (τ, K, Q)-spanning sets is guaranteed by property (ii); indeed, $\mathcal{U}$ is a (τ, K, Q)-spanning set for every $\tau > 0$. A pair of the form (Q, Q) is admissible if and only if Q is a compact and controlled invariant set.

Definition 1.4.2. *Given an admissible pair (K, Q), we define the* **invariance entropy of** (K, Q) *by*

$$h_{\text{inv}}(K, Q) = h_{\text{inv}}(K, Q; \Sigma) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{inv}}(\tau, K, Q).$$

Here, we use the convention that $\log = \log_e = \ln$. If $K = Q$, again we omit the argument K and write $h_{\text{inv}}(Q)$. Moreover, we let $\log \infty := \infty$.

Hence, invariance entropy is a nonnegative (possibly infinite) quantity which is assigned to an admissible pair (K, Q). In fact, the invariance entropy of (K, Q) measures the exponential growth rate of the minimal number of different control functions sufficient to stay in Q when starting in K, as time tends to infinity.

The following proposition presents some basics properties of $r_{\text{inv}}(\tau, K, Q)$ and $h_{\text{inv}}(K, Q)$ with respect to their arguments, including the finiteness of them.

Proposition 1.4.3. *Let (K, Q) be an admissible pair. Then the following assertions hold:*

i) If $\tau_1 < \tau_2$, then $r_{\text{inv}}(\tau_1, K, Q) \leq r_{\text{inv}}(\tau_2, K, Q)$.

ii) If $Q \subset P$, then (K, P) is admissible and $r_{\text{inv}}(\tau, K, Q) \geq r_{\text{inv}}(\tau, K, P)$ which implies that $h_{\text{inv}}(K, Q) \geq h_{\text{inv}}(K, P)$.

iii) If $L \subset K$ is closed, then (L, Q) is admissible and $r_{\text{inv}}(\tau, L, Q) \leq r_{\text{inv}}(\tau, K, Q)$ which implies that $h_{\text{inv}}(L, Q) \leq h_{\text{inv}}(K, Q)$.

iv) If $\Sigma' = (\mathbb{R}, M, U, \mathcal{U}', \varphi')$ is another control system on M such that $\mathcal{U}' \supset \mathcal{U}$ and $\varphi'(t, x, \omega) = \varphi(t, x, \omega)$ for all $\omega \in \mathcal{U}$, then (K, Q) is also admissible for Σ' and $h_{\text{inv}}(K, Q; \Sigma') \leq h_{\text{inv}}(K, Q; \Sigma)$.

v) If Q is open, then $r_{\text{inv}}(\tau, K, Q)$ is finite for all $\tau > 0$.

vi) If Q is a compact controlled invariant set, then:

vi1) The number $r_{\text{inv}}(\tau, Q)$ is either finite for all $\tau > 0$ or for none.

vi2) The function $\log r_{\text{inv}}(\cdot, Q) : (0, +\infty) \to \mathbb{R}_+ \cup \{\infty\}$, $\tau \mapsto \log r_{\text{inv}}(\tau, Q)$, is subaditive and therefore

$$h_{\text{inv}}(Q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{inv}}(\tau, Q) = \inf_{\tau > 0} \frac{1}{\tau} \log r_{\text{inv}}(\tau, Q).$$

Another notion of entropy (whose definition requires a metric) associated with an admissible pair is given in sequence.

Definition 1.4.4. *Given an admissible pair (K, Q) such that Q is closed in M, and a metric ϱ on M, we define the* **outer invariance entropy** *of (K, Q) by*

$$h_{\text{inv,out}}(K, Q) := h_{\text{inv,out}}(K, Q; \varrho; \Sigma) := \lim_{\varepsilon \searrow 0} h_{\text{inv}}(K, N_\varepsilon(Q)) = \sup_{\varepsilon > 0} h_{\text{inv}}(K, N_\varepsilon(Q)),$$

where $N_\varepsilon(Q) = \{y \in M; \exists\, x \in Q \text{ with } d(x, y) < \varepsilon\}$ denotes the ε-neighborhood of Q.

These two quantities relate as follows

$$0 \leq h_{\text{inv,out}}(K, Q) \leq h_{\text{inv}}(K, Q) \leq \infty.$$

Although in general these quantities do not coincide, this fact is verified (under some assumption which we expose in sequence) in the case of linear control systems (see [27, Corollary 5.3]):

Theorem 1.4.5. *Consider a linear control system Σ_{lin} given by the differential equation*

$$\dot{x}(t) = Ax(t) + B\omega(t), \ \ \omega \in \mathcal{U},$$

where the matrix pair (A, B) is controllable and such that A has no eigenvalues on the imaginary axis (that is, A is hyperbolic). Further assume that the control range U is a compact and convex set with $0 \in \text{int}U$. Let $D \subset \mathbb{R}^d$ be the unique control set for Σ_{lin} with nonempty interior. Then for every compact set $K \subset D$ it holds that

$$h_{\text{inv}}(K, Q) \leq \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda \text{Re}(\lambda)\},$$

where $\sigma(A)$ denotes the spectrum of A and n_λ is the multiplicity of $\lambda \in \sigma(A)$. If, additionally, K has positive Lebesgue measure and $Q := \overline{D}$ it holds that

$$h_{\text{inv}}(K, Q) = h_{\text{inv,out}}(K, Q) = \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda \text{Re}(\lambda)\}.$$

Remark 1.4.6. *Note that the definitions of $h_{\text{inv}}(K, Q)$ and $h_{\text{inv,out}}(K, Q)$ can be adapted for discrete-time control systems as presented in [27].*

From now, consider a discrete-time control system $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$. We say that a compact subset $Q \subset X$ with nonempty interior is **strongly invariant** for Σ if for every $x \in Q$, there is $u \in U$ such that $F(x, u) \in \text{int}Q$, or equivalently, $\varphi(1, x, u) \in \text{int}Q$.

Definition 1.4.7. *A triple $\mathcal{C} = (\mathcal{A}, \tau, G)$ is called an* **invariant open cover** *of Q if it satisfies the following properties:*

- *$\mathcal{A}$ is an open cover of Q;*
- *τ is a positive integer;*
- *G is a finite sequence of maps $G_k : \mathcal{A} \to U$, $k = 0, 1, \cdots, \tau - 1$ such that for each $A \in \mathcal{A}$ it holds that*

$$\varphi(k, A, G(A)) \subset \text{int}Q, \quad \textit{for } k = 1, \cdots, \tau,$$

that is, if the initial value $x \in Q$ lies in the set $A \in \mathcal{A}$, then any control $\omega = (u_k)_{k \in \mathbb{N}_0}$ which satisfies $u_k = G_k(A)$ for $k = 0, 1, \cdots, \tau - 1$ yields $\varphi(k, x, \omega) \in \text{int}Q$, for $k = 1, \cdots, \tau$.

It is possible to verify that all strongly invariant sets Q for $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$ admit an invariant open cover (see [27, pg. 69-70]).

Given an arbitrary invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$, for any sequence $\alpha = \{A_i\}_{i \in \mathbb{N}_0}$ of sets in $\mathcal{A}$, we define the control sequence

$$\omega(\alpha) := (u_0, u_1, \cdots) \text{ with } (u_l)_{l=(i-1)\tau}^{i\tau - 1} = G(A_{i-1}), \quad \text{for all } i \geq 1, \tag{1.4-6}$$

that is,

$$\omega(\alpha) = (\underbrace{u_0, \cdots, u_{\tau-1}}_{G(A_0)}, \underbrace{u_\tau, \cdots, u_{2\tau-1}}_{G(A_1)}, \cdots).$$

We further define for each $j \in \mathbb{N}$ the set

$$B_j(\alpha) := \{x \in X;\ \varphi(i\tau, x, \omega(\alpha)) \in A_i, \text{ for } i = 0, 1, \cdots, j - 1\}. \tag{1.4-7}$$

Then $B_j(\alpha)$ is an open set, since it can be written as the finite intersection of preimages of open sets under continuous maps, namely

$$B_j(\alpha) = \bigcap_{i=0}^{j-1} \{x \in X;\ \varphi(i\tau, x, \omega(\alpha)) \in A_i\} = \bigcap_{i=0}^{j-1} \varphi_{i\tau, \omega(\alpha)}^{-1}(A_i).$$

Furthermore, for each $j \in \mathbb{N}$, letting α run through all sequences of elements in $\mathcal{A}$, the family

$$\mathcal{B}_j = \mathcal{B}_j(\mathcal{C}) := \{B_j(\alpha);\ \alpha \in \mathcal{A}^{\mathbb{N}_0}\}$$

is an open cover of Q.

Let $N(\mathcal{B}_j; Q)$ denote the minimal number of elements in a finite subcover of $\mathcal{B}_j$. Then, the **topological feedback entropy** $h_{fb}(Q)$ is defined by

$$h_{\text{fb}}(\mathcal{C}) = h_{\text{fb}}(\mathcal{C}; \Sigma) := \lim_{j\to\infty} \frac{1}{j\tau} \log N(\mathcal{B}_j; Q), \tag{1.4-8}$$

$$h_{\text{fb}}(Q) = h_{\text{fb}}(Q; \Sigma) := \inf_{\mathcal{C}} h_{\text{fb}}(\mathcal{C}; \Sigma),$$

where the infimum is taken over all invariant open covers $\mathcal{C} = (\mathcal{A}, \tau, G)$ of Q.

Next we present the computation of a non smooth discrete-time control system contained in [32].

Example 1.4.8. *Consider the control system $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$ where the right-hand side $F : \mathbb{R} \times U \to \mathbb{R}$ is defined by $F(x, u) = \max\left\{\frac{2x-1}{2}, \frac{1-x}{2}\right\} + u$ and $U \subset \mathbb{R}$ is such that $U \ni 0$. Then the set $Q = [0, 1]$ is strongly invariant, because $F(Q, 0) \subset \text{int}Q$. Let $\mathcal{C} = (\mathcal{A}, \tau, G)$, where $\mathcal{A} = \{A := (-\frac{1}{3}, \frac{4}{3})\}$, $\tau \in \mathbb{N}$ and $G \equiv 0$. Since $F(Q, 0) \subset \text{int}Q$, then $\mathcal{C}$ is an invariant open cover. In this case, we have $\mathcal{A}^{\mathbb{N}_0}$ admits only one element $\alpha : \mathbb{N}_0 \to \mathcal{A}$, $\alpha_n = A$ for all $n \in \mathbb{N}_0$.*

Note that

$$B_j(\alpha) = B_1(\alpha) = A \supset Q, \text{ for all } j \in \mathbb{N},$$

because $F(A, 0) \subset Q$. Then $\mathcal{B}_j = \{A\}$ and hence $N(\mathcal{B}_j; Q) = 1$. Therefore

$$h_{\text{fb}}(\mathcal{C}) := \lim_{j\to\infty} \frac{1}{j\tau} \log N(\mathcal{B}_j; Q) = \lim_{j\to\infty} \frac{1}{j\tau} \log 1 = 0$$

and we obtain

$$0 \leq h_{fb}(Q) \leq h_{\text{fb}}(\mathcal{C}) = 0,$$

that is $h_{\text{fb}}(Q) = 0$.

In order to show that the limit (1.4-8) exists, the following general lemma is necessary:

Lemma 1.4.9. *Let $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ and $f : \mathbb{T}_+ \to \mathbb{R}$ be a subadditive function, that is, $f(t+s) \leq f(t) + f(s)$, for all $s, t \in \mathbb{T}_+$. Suppose further that f is bounded from above on an interval of the form $\mathbb{T} \cap [0, t_0]$ with $t_0 > 0$. Then $\lim_{t\to\infty} f(t)/t$ converges (the limit may be $-\infty$), and*

$$\lim_{t\to\infty} \frac{f(t)}{t} = \inf_{t>0} \frac{f(t)}{t} =: \gamma.$$

Proof. See Lemma B.3 of [27] for the proof. □

In the following, we introduce a modified version of invariance entropy for discrete-time control systems, which turns out to coincide with the topological feedback entropy.

Definition 1.4.10. *Consider $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$ and a compact strongly invariant set $Q \subset X$ for Σ with $\mathrm{int}Q \neq \emptyset$. For $\tau \in \mathbb{N}$, a subset $\mathcal{S} \subset \mathcal{U}$ is called $(\tau, Q, \mathrm{int}Q)-$**spanning** if*

$$\forall x \in Q,\ \exists \omega \in \mathcal{S} \ \text{ such that } \varphi([1,\tau], x, \omega) \subset \mathrm{int}Q.$$

The minimal cardinality of such a set is denoted by $r_{\text{inv,int}}(\tau, Q)$ and the **inner invariance entropy** *of Q is defined by*

$$h_{\text{inv,int}}(Q) = h_{\text{inv,int}}(Q; \Sigma) := \lim_{\tau\to\infty} \frac{1}{\tau} \log r_{\text{inv,int}}(\tau, Q).$$

Since the sequence $\tau \mapsto \log r_{\text{inv,int}}(\tau, Q)$ is subadditive, the definition of $h_{\text{inv,int}}(Q)$ is correct and

$$h_{\text{inv,int}}(Q) = \lim_{\tau\to\infty} \frac{1}{\tau} \log r_{\text{inv,int}}(\tau, Q) = \inf_{\tau\geq 1} \frac{1}{\tau} \log r_{\text{inv,int}}(\tau, Q).$$

The following result, whose proof can be found in [10], relates the quantities $h_{\text{fb}}(Q)$ and $h_{\text{inv,int}}(Q)$.

Theorem 1.4.11. *Given a strongly invariant compact set $Q \subset X$ for $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$, we have*

$$h_{\text{fb}}(Q) = h_{\text{inv,int}}(Q).$$

CHAPTER 2

INVARIANCE PRESSURE FOR CONTINUOUS-TIME CONTROL SYSTEMS

The main notion of this thesis - invariance pressure - is presented in this chapter for continuous-time control systems $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$. The original notion was introduced by Colonius, Cossich and Santana in [6] and generalized by the same authors in [7] for admissible pairs. Roughly speaking, the invariance pressure of a Σ measures the weighted information necessary that open loop controls provide to the Σ in order to keep it in a given subset Q of the state space, starting from a compact set $K \subset Q$. The term "weighted information" can express the physical quantities (see Examples 2.1.6 and 3.3.8) depending on the considered weight f (which is a continuous function defined on the control-valued-space U) and the control functions.

Given an admissible pair (K, Q), if we denote by $a_\tau(f, K, Q)$ the weighted information (with weight f) necessary, produced by control functions, for that every trajectory of Σ starting in K remain in the bigger set Q up to time τ, then the invariance pressure $P_{\text{inv}}(f, K, Q)$ is the exponential growth rate of $a_\tau(f, K, Q)$ as τ tends to infinity,

$$P_{\text{inv}}(f, K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q).$$

This definition is similar to that of topological pressure which measures the "weighted" exponential orbit complexity of a dynamical system. It was introduced by the physical-mathematician David Ruelle in 1967 (see [34]) and generalized by Peter Walters in 1976 for (discrete-time) dynamical systems on a compact metric space (see [38]). Despite the suggestive name, the definition of invariance pressure is not necessarily related to pressure, in the physical sense. This name was given in order to make a parallel with the dynamical concept of topological pressure.

In this chapter we follow the ideas presented in [6], [7], [9], Kawan [24], [27] and [39]. It is organized in the following way: Section 2.1 is presented the definition of invariance pressure and outer invariance pressure for continuous-time control systems as well as the basic properties of these amounts such as the finiteness of $P_{\text{inv}}(f, K, Q)$, the Lipschitz property of $P_{\text{inv}}(\cdot, K, Q)$ and how it behaves in relation to its arguments. Finally, in Section 2.2 we explore some elementary properties of invariance pressure which are similar to the well-known properties of the classical topological pressure notions in dynamical systems, for example, the time discretization, the power rule, the product rule and the invariance under conjugacy.

2.1 Definitions and Basic Properties

Considering a continuous-time control system Σ on a connected smooth manifold M, in this section we define the invariance pressure and the outer invariance pressure of Σ and we explore some basic properties that this quantities satisfy.

Initially, consider an admissible pair (K, Q) and denote by $C(U, \mathbb{R})$ the set of all continuous functions $f : U \to \mathbb{R}$. Moreover, given $\tau > 0$, $\omega \in \mathcal{U}$ and a $f \in C(U, \mathbb{R})$ we denote by $(S_\tau f)(\omega)$ the real number $\int_0^\tau f(\omega(s))ds$. Then we can define

$$a_\tau(f, K, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \text{ is a } (\tau, K, Q)\text{-spanning set} \right\}. \qquad (2.1\text{-}1)$$

Note that in the definition of $a_\tau(f, K, Q)$ it suffices to take the infimum over those (τ, K, Q)-spanning sets which do not have proper subsets that are also (τ, K, Q)-spanning sets. In fact, if $\mathcal{S}$ is a (τ, K, Q)-spanning set which contains another (τ, K, Q)-spanning $\mathcal{S}'$, the summands in $\mathcal{S} \setminus \mathcal{S}'$ can be omitted, since $e^{(S_\tau f)(\omega)} > 0$.

Moreover, we can observe that since U is compact, then

$$a_\tau(f,K,Q) = \inf_{\mathcal{S}} \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} \geq e^{\tau \inf f} \inf_{\mathcal{S}} (\#\mathcal{S}) = e^{\tau \inf f} r_{\text{inv}}(K,Q) > 0.$$

Definition 2.1.1. *The* **invariance pressure** *of $P_{\text{inv}}(f,K,Q)$ of the admissible pair (K,Q) with respect to f is given by*

$$P_{\text{inv}}(f,K,Q) = P_{\text{inv}}(f,K,Q;\Sigma) := \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f,K,Q).$$

Hence, the invariance pressure is defined as the exponential growth rate of the minimal weighted information (with weight f) that control functions produce in order to keep every trajectory starting in K in the bigger set Q up to time τ, as τ tends to infinity.

If $K = Q$ we omit the argument K and write $a_\tau(f,Q)$ and $P_{\text{inv}}(f,Q)$. Note that, in this case, we assume that Q is compact and controlled invariant.

Definition 2.1.2. *Given an admissible pair (K,Q) such that Q is closed in M, and a metric ϱ on M, we define the* **outer invariance pressure** *of (K,Q) with respect to f by*

$$P_{\text{inv,out}}(f,K,Q) = P_{\text{inv,out}}(f,K,Q;\varrho;\Sigma) := \lim_{\varepsilon\searrow 0} P_{\text{inv}}(f,K,N_\varepsilon(Q)),$$

Note that the limit for $\varepsilon \searrow 0$ exists and equals the supremum over $\varepsilon > 0$, since from Proposition 2.1.7 it follows that the pairs $(K, N_\varepsilon(Q))$ are admissible and that $\varepsilon_1 < \varepsilon_2$ implies $P_{\text{inv}}(f,K,N_{\varepsilon_1}(Q)) \geq P_{\text{inv}}(f,K,N_{\varepsilon_2}(Q))$. Furthermore

$$-\infty < P_{\text{inv,out}}(f,K,Q) \leq P_{\text{inv}}(f,K,Q) \leq \infty$$

for every admissible pair (K,Q) and $f \in C(U,\mathbb{R})$.

These definitions deserve several comments. First observe that $P_{\text{inv}}(f,K,Q) \geq 0$ for $f \geq 0$.

If $f = \mathbf{0}$ is the null function in $C(U, \mathbb{R})$, then $\sum_{\omega \in \mathcal{S}} e^{(S_\tau \mathbf{0})(\omega)} = \sum_{\omega \in \mathcal{S}} 1 = \#\mathcal{S}$, hence

$$\begin{aligned}
a_\tau(\mathbf{0}, K, Q) &= \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_\tau \mathbf{0})(\omega)};\ \mathcal{S} \text{ is a } (\tau, K, Q)\text{-spanning set} \right\} \\
&= \inf \left\{ \#\mathcal{S};\ \mathcal{S} \text{ is a } (\tau, K, Q)\text{-spanning set} \right\} \\
&= r_{\text{inv}}(\tau, K, Q). \qquad (2.1\text{-}2)
\end{aligned}$$

Taking the logarithm, dividing by τ and letting τ tend to ∞ one finds that $P_{\text{inv}}(\mathbf{0}, K, Q) = h_{\text{inv}}(K, Q)$. Hence the (outer) invariance pressure generalizes the (outer) invariance entropy.

Another comment concerns the independence of the invariance pressure with respect to uniformly equivalent metrics. We say that two metrics ϱ_1 and ϱ_2 on M are **uniformly equivalent** on Q, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in Q$ and for all $y \in M$ with $\varrho_i(x, y) < \delta$ implies that $\varrho_j(x, y) < \varepsilon$, for $i, j = 1, 2$, $i \neq j$.

The following proposition states that the value of the outer invariance pressure of (K, Q) does not change when we consider uniformly equivalent metrics. Since the proof is similar to Kawan [27, Proposition 2.5], we will omit it.

Proposition 2.1.3. *Let (K, Q) be an admissible pair such that Q is closed in M. If ϱ_1 and ϱ_2 are two metrics on M which are uniformly equivalent on Q, then $P_{\text{inv,out}}(f, K, Q; \varrho_1) = P_{\text{inv,out}}(f, K, Q; \varrho_2)$ for all $f \in C(U, \mathbb{R})$. If Q is compact, then this is automatically satisfied, and in this case the outer invariance pressure is independent of the metric.*

The next proposition shows that we just need finite spanning sets to get $a_\tau(f, K, Q)$.

Proposition 2.1.4. *Consider an admissible pair (K, Q) with Q open in M and $f \in C(U, \mathbb{R})$. Then*

$$a_\tau(f, K, Q) = \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \textit{ is a finite } (\tau, K, Q)\textit{-spanning set} \right\}.$$

Proof. First we show that if $\mathcal{S}$ is a (τ, K, Q)-spanning set, $\tau > 0$, then there exists a finite (τ, K, Q)-spanning set $\mathcal{S}' \subset \mathcal{S}$. In fact, take an arbitrary $x \in K$. Since $\mathcal{S}$ is (τ, K, Q)-spanning, there is $\omega_x \in \mathcal{S}$ with $\varphi(t, x, \omega_x) \in Q$ for $t \in [0, \tau]$. Openness of Q and uniform continuity of $\varphi(t, \cdot, \omega)$ in $t \in [0, \tau]$, we find an open neighborhood W_x of x such that $\varphi([0, \tau], W_x, \omega_x) \subset Q$. Compactness of K implies the existence of $x_1, \ldots, x_k \in K$ such that $K \subset \bigcup_{i=1}^{k} W_{x_i}$. It is easy to see that the set $\mathcal{S}' = \{\omega_{x_1}, \ldots, \omega_{x_k}\} \subset \mathcal{S}$ is a (τ, K, Q)-spanning set.

Now define

$$\widetilde{a}_\tau(f, K, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \text{ is a finite } (\tau, K, Q)\text{-spanning set} \right\}.$$

Since clearly $a_\tau(f, K, Q) \leq \widetilde{a}_\tau(f, K, Q)$, we just have to prove the reverse inequality. Given a (τ, K, Q)-spanning set $\mathcal{S}$, as shown earlier there is a finite (τ, K, Q)-spanning subset $\mathcal{S}' \subset \mathcal{S}$. Hence $\sum_{\omega \in \mathcal{S}'} e^{(S_\tau f)(\omega)} \leq \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)}$, which implies that $\widetilde{a}_\tau(f, K, Q) \leq a_\tau(f, K, Q)$. □

Remark 2.1.5. *Since for the outer invariance entropy one considers $(\tau, K, N_\varepsilon(Q))$-spanning sets, $\varepsilon > 0$, by Proposition 2.1.4 it is sufficient to consider finite $(\tau, K, N_\varepsilon(Q))$-spanning sets, because of the openness of $N_\varepsilon(Q)$.*

Example 2.1.6. *Assume that $f \in C(U, \mathbb{R})$ and that $\mathcal{O}^+(x) \subset Q$, for all $x \in K$, that is, the solutions always remains in Q when starting in K. We show that $P_{\text{inv}}(f, K, Q) = \inf f$. Since for every (τ, K, Q)-spanning set $\mathcal{S}$ the estimate*

$$\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \geq e^{\tau \inf f} \cdot \#\mathcal{S} \geq e^{\tau \inf f}$$

holds, it follows that $P_{\text{inv}}(f, K, Q) \geq \inf f$. Conversely, given $\varepsilon > 0$ there exists $u \in U$ with

$$f(u) \leq \inf f + \varepsilon.$$

Then the one-point set $\mathcal{S} = \{\omega\}$, where $\omega(t) \equiv u$, is (τ, K, Q)-spanning and

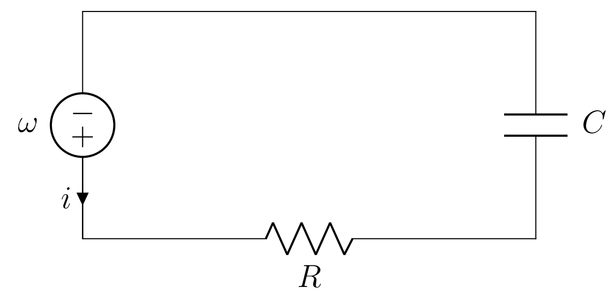
$$\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} = e^{(S_\tau f)(\omega)} = e^{\tau f(u)} \leq e^{\tau \inf f + \tau\varepsilon}.$$

Taking the infimum over all (τ, K, Q)-spanning sets one finds that the invariance pressure satisfies

$$P_{\text{inv}}(f, K, Q) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q) \leq \limsup_{\tau \to \infty} \frac{1}{\tau} \log e^{\tau \inf f} + \varepsilon = \inf f + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $P_{\text{inv}}(f, K, Q) \leq \inf f$.

A simple illustration of this case is represented in the above figure where we consider a RC-circuit driven by a voltage source $\omega(t) \in U := [a, b]$, $0 \leq a < b$.



Since the current i is in the direction indicated, then there will be a drop in the voltage across each of the elements. By Kirchhoff's law, the total voltage drop across these elements must be balanced by that supplied by the voltage source. So if the voltage across the resistor and capacitor at time t are $\omega_R(t)$, $\omega_C(t)$, respectively, we have

$$\omega(t) - \omega_R(t) - \omega_C(t) = 0.$$

But if the charge on the capacitor is $q(t)$, the resistance of the resistor is $R > 0$ and the capacitance of the capacitor is $C > 0$, then

$$\omega_R(t) = Ri(t), \;\; \omega_C(t) = q(t)/C, \;\; i(t) = \dot{q}(t), \;\; t \geq 0.$$

Hence we obtain the following first order differential equation

$$R\dot{q}(t) + \frac{1}{C}q(t) = \omega(t).$$

Then the set $Q := [aC, bC]$ satisfies $\mathcal{O}^+(x) \subset Q$, for all $x \in Q$. In particular, if the current $i > 0$ is constant in time and $f(u) = iu$, then $(S_\tau f)(\omega)$ represents the electrical energy spent between the instants 0 and τ. If fact, if $P(t)$ and $E(t)$ denote the electric power and the electrical energy transferred at time t, respectively, then

$$(S_\tau f)(\omega) = \int_0^\tau i\omega(t)dt = \int_0^\tau P(t)dt = E(\tau) - E(0).$$

In this case, the information measured by the invariance pressure is the variation of electrical energy, that is, $P_{\text{inv}}(f, Q)$ measures the exponential growth rate of the minimal amount of total variation of electrical energy produced by the voltage source to keep the charge in Q as time *tends to infinity. In this case,* $P_{\text{inv}}(f, Q) = \inf f = ia$.

The next results of this section show several basic properties of the invariance pressure with respect to an admissible pair.

Proposition 2.1.7. *For a control system* $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ *and an admissible pair* (K, Q)*, the following assertions hold:*

i) If $0 < \tau_1 < \tau_2$ *and* $f \geq 0$*, then* $a_{\tau_1}(f, K, Q) \leq a_{\tau_2}(f, K, Q)$*;*

ii) If $Q \subset R$*, then* (K, R) *is admissible and* $a_\tau(f, K, Q) \geq a_\tau(f, K, R)$*; hence* $P_{\text{inv}}(f, K, Q) \geq P_{\text{inv}}(f, K, R)$*;*

iii) If $L \subset K$ *is closed in* M*, then* (L, Q) *is admissible and* $a_\tau(f, L, Q) \leq a_\tau(f, K, Q)$*; hence* $P_{\text{inv}}(f, L, Q) \leq P_{\text{inv}}(f, K, Q)$*;*

iv) If $\Sigma' = (\mathbb{R}, M, U, \mathcal{U}', \varphi')$ *is another control system such that set of admissible control functions* $\mathcal{U}'$ *contain* $\mathcal{U}$ *and* $\varphi'(t, x, \omega) = \varphi(t, x, \omega)$ *whenever* $\omega \in \mathcal{U}$*, then* (K, Q) *is also admissible for* Σ' *and* $P_{\text{inv}}(f, K, Q; \Sigma') \leq P_{\text{inv}}(f, K, Q; \Sigma)$*. In particular, if* $\Sigma' = (\mathbb{R}, M, V, \mathcal{V}, \varphi')$ *is such that* $V \subset U$*,* $\mathcal{V} := \{\omega \in \mathcal{U};\ \omega(t) \in V \text{ a.e.}\}$*,* $\varphi'(t, x, \omega) = \varphi(t, x, \omega)$ *whenever* $\omega \in \mathcal{V}$ *and* (K, Q) *is admissible for* Σ'*, then* $P_{\text{inv}}(f, K, Q; \Sigma) \leq P_{\text{inv}}(f, K, Q; \Sigma')$*.*

The previous properties contained in Proposition 2.1.7 are easy to see and hence we will not present their proofs.

In order to expose some properties of the function $P_{\text{inv}}(\cdot, K, Q)$, we will need the following elementar lemma.

Lemma 2.1.8. *Let* $a_i \geq 0, b_i > 0, i = 1, .., n \in \mathbb{N}$*, be real numbers. Then*

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \geq \min_{i=1,\ldots,n} \left(\frac{a_i}{b_i}\right).$$

Proof. Let $n = 2$. Then we may assume that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$. Dividing numerator and denominator by b_1 one can further assume that $b_1 = 1$, hence the assumption takes the form $a_1 \leq \frac{a_2}{b_2}$ and the assertion reduces to $\frac{a_1+a_2}{1+b_2} \geq a_1$. This is equivalent to

$$a_1 + a_2 \geq a_1 + a_1 b_2, \text{ i.e., } a_2 \geq a_1 b_2,$$

which is our assumption. The induction step from n to $n+1$ follows since

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} = \frac{\sum_{i=1}^{n} a_i + a_{n+1}}{\sum_{i=1}^{n} b_i + b_{n+1}} \geq \min\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}, \frac{a_{n+1}}{b_{n+1}}\right) \geq \min_{i=1,\ldots,n+1}\left(\frac{a_i}{b_i}\right).$$

□

Corollary 2.1.9. *Consider real numbers $a_i \geq 0$, $b_i > 0$, $i \in \mathbb{N}$. Then*

$$\frac{\sum_{i\in\mathbb{N}} a_i}{\sum_{i\in\mathbb{N}} b_i} \geq \inf_{i\in\mathbb{N}}\left(\frac{a_i}{b_i}\right).$$

Proof. By Lemma 2.1.8, for each $n \in \mathbb{N}$ we have

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \geq \min_{i=1,\ldots,n}\left(\frac{a_i}{b_i}\right) \geq \inf_{i\in\mathbb{N}}\left(\frac{a_i}{b_i}\right).$$

The result follows if we take the limit for $n \to \infty$. □

The following proposition present a sufficient condition for take the infimum $a_\tau(f, K, Q)$ over countable (τ, K, Q)-spanning sets.

Proposition 2.1.10. *If (K, Q) is an admissible pair and $f \in C(U, \mathbb{R})$ such that $P_{\text{inv}}(f, K, Q) < \infty$, then for all $\tau > 0$,*

$$a_\tau(f, K, Q) := \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \textit{ is a countable } (\tau, K, Q)\textit{-spanning set}\right\}.$$

Proof. Firstly note that if $P_{\text{inv}}(f, K, Q) < \infty$, $f \in C(U, \mathbb{R})$, then for all $\tau > 0$ there is a countable (τ, K, Q)-spanning set $\mathcal{S}$. In fact, if there exists $\tau > 0$ such that all (τ, K, Q)-spanning is uncountable, then all (σ, K, Q)-spanning set with $\sigma \geq \tau$ is also uncountable, because all (σ, K, Q)-spanning set is a (τ, K, Q)-spanning. Hence we should have $a_\tau(f, K, Q) = \infty$ which implies that $P_{\text{inv}}(f, K, Q) = \infty$.

Secondly observe also that if the invariance pressure with respect to $f \in C(U, \mathbb{R})$ is finite, then there are for every $\tau > 0$ countable (τ, K, Q)-spanning sets $\mathcal{S}$ with

$$\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} < \infty,$$

because any sum over uncountably many positive numbers must be equal to $+\infty$.

Now put

$$\hat{a}_\tau(f,K,Q) := \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \text{ is a countable } (\tau,K,Q)\text{-spanning set}\right\}.$$

It is clear that $a_\tau(f,K,Q) \le \hat{a}_\tau(f,K,Q)$. To show the reverse inequality, suppose that $\hat{a}_\tau(f,K,Q) > a_\tau(f,K,Q)$. Then given $\varepsilon := \hat{a}_\tau(f,K,Q) - a_\tau(f,K,Q)$ there exists a (τ,K,Q)-spanning set such that

$$a_\tau(f,K,Q) \le \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} < a_\tau(f,K,Q) + \varepsilon = \hat{a}_\tau(f,K,Q). \tag{2.1-3}$$

Note that by (2.1-3) $\mathcal{S}$ can not be countable. Hence $\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} = +\infty$ witch implies that $\hat{a}_\tau(f,K,Q) = +\infty$. Since $P_{\text{inv}}(f,K,Q) < \infty$, there exists a countable (τ,K,Q)-spanning set $\hat{\mathcal{S}}$ with $\sum_{\omega\in\hat{\mathcal{S}}} e^{(S_\tau f)(\omega)} < \infty$ and this contradicts $\hat{a}_\tau(f,K,Q) \le \sum_{\omega\in\hat{\mathcal{S}}} e^{(S_\tau f)(\omega)}$. Hence we have $\hat{a}_\tau(f,K,Q) = a_\tau(f,K,Q)$. □

Proposition 2.1.11. *The following assertions hold for an admissible pair (K,Q), functions $f,g \in C(U,\mathbb{R})$ and $c \in \mathbb{R}$:*

i) If $f \le g$, then $P_{\text{inv}}(f,K,Q) \le P_{\text{inv}}(g,K,Q)$. In particular $h_{\text{inv}}(K,Q) + \inf f \le P_{\text{inv}}(f,K,Q) \le h_{\text{inv}}(K,Q) + \sup f$;

ii) $P_{\text{inv}}(f+c,K,Q) = P_{\text{inv}}(f,K,Q) + c$;

Proof. i) If $f \le g$, it follows that $\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} \le \sum_{\omega\in\mathcal{S}} e^{(S_\tau g)(\omega)}$ for all (τ,K,Q)-spanning sets $\mathcal{S}$, because the exponential function is increasing. Hence $a_\tau(f,K,Q) \le a_\tau(g,K,Q)$ and so $P_{\text{inv}}(f,K,Q) \le P_{\text{inv}}(g,K,Q)$.

ii) One finds that

$$\begin{aligned} a_\tau(f+c,K,Q) &= \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau(f+c))(\omega)};\ \mathcal{S} \text{ is a } (\tau,K,Q)\text{-spanning}\right\} \\ &= \inf\left\{e^{\tau c}\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \text{ is a } (\tau,K,Q)\text{-spanning}\right\} \\ &= e^{\tau c} a_\tau(f,K,Q), \end{aligned}$$

hence

$$
\begin{aligned}
P_{\text{inv}}(f+c,K,Q) &= \lim_{\tau\to\infty}\frac{1}{\tau}\log a_\tau(f+c,K,Q) = \lim_{\tau\to\infty}\frac{1}{\tau}\log\left(e^{\tau c}a_\tau(f,K,Q)\right)\\
&= c + P_{\text{inv}}(f,K,Q).
\end{aligned}
$$

□

Remark 2.1.12. *By item (i) of Proposition 2.1.11 we get that for each $f \in C(U,\mathbb{R})$, $P_{\text{inv}}(f,K,Q) = \infty$ if, and only if, $h_{\text{inv}}(K,Q) = \infty$. Hence we can change the assumption $P_{\text{inv}}(f,K,Q) < \infty$ in Proposition 2.1.10 for $h_{\text{inv}}(K,Q) < \infty$.*

The next result presents a regularity property of the map $P_{\text{inv}}(\cdot,K,Q) : C(U,\mathbb{R}) \to \mathbb{R}$.

Proposition 2.1.13. *If (K,Q) is an admissible pair with $h_{\text{inv}}(K,Q) < \infty$, then*

$$
|P_{\text{inv}}(f,K,Q) - P_{\text{inv}}(g,K,Q)| \le \|f-g\|_\infty,
$$

where $\|\cdot\|_\infty$ is the uniform norm on $C(U,\mathbb{R})$.

Proof. Note that using Corollary 2.1.9 for the second inequality below, one finds

$$
\begin{aligned}
\frac{a_\tau(g,K,Q)}{a_\tau(f,K,Q)} &= \frac{\inf_{\mathcal{S}}\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau g)(\omega)}\right\}}{\inf_{\mathcal{S}}\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}\right\}} \ge \inf_{\mathcal{S}}\left\{\frac{\sum_{\omega\in\mathcal{S}} e^{(S_\tau g)(\omega)}}{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}}\right\}\\
&\ge \inf_{\mathcal{S}}\left\{\inf_{\omega\in\mathcal{S}}\frac{e^{(S_\tau g)(\omega)}}{e^{(S_\tau f)(\omega)}}\right\} \ge e^{-\tau\|f-g\|_\infty},
\end{aligned}
$$

where the infimum above are taken over all countable (τ,K,Q)-spanning sets. Therefore $\frac{a_\tau(f,K,Q)}{a_\tau(g,K,Q)} \le e^{\tau\|f-g\|_\infty}$ and so

$$
\begin{aligned}
P_{\text{inv}}(f,K,Q) - P_{\text{inv}}(g,K,Q) &= \lim_{\tau\to\infty}\frac{1}{\tau}\log\frac{a_\tau(f,K,Q)}{a_\tau(g,K,Q)} \le \lim_{\tau\to\infty}\frac{1}{\tau}\log e^{\tau\|f-g\|_\infty}\\
&= \|f-g\|_\infty.
\end{aligned}
$$

Interchanging the roles of f and g one finds assertion. □

Open Question 1. *In [40], Walters present several necessary and sufficient conditions for the topological pressure of a continuous transformation on a compact metric space to be differen-*

tiable, in the Fréchet sense, with respect to the potentials f defined on the state space. Is the map $P_{\text{inv}}(\cdot, K, Q) : C(U, \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ differentiable in some sense?

Open Question 2. *In [39, Theorem 9.7 (v)] we can see that the topological pressure is convex as function of the potentials. Is this property verified in the case of invariance pressure?*

The next corollaries deal with the finiteness of invariance pressure.

Corollary 2.1.14. *Consider $f \in C(U, \mathbb{R})$. Then*

i) If Q is open, then $a_\tau(f, K, Q)$ is finite for all $\tau > 0$;

ii) If Q is a compact controlled invariant set, then $a_\tau(f, Q)$ is either finite for all $\tau > 0$ or for none.

Proof. The two statements follows from the inequalities

$$e^{\tau \inf f} r_{\text{inv}}(\tau, K, Q) \leq a_\tau(f, K, Q) \leq e^{\tau \sup f} r_{\text{inv}}(\tau, K, Q)$$

and itens (v) and (vi) of Proposition 1.4.3 . □

Remark 2.1.15. *Note that Propositions 2.1.11, 2.1.13 and Corollary 2.1.14(i) also hold for the outer invariance pressure.*

As an immediate consequence, we have:

Corollary 2.1.16. *If $f \in C(U, \mathbb{R})$ and Q is compact controlled invariant, then the following assertions are equivalent:*

i) $P_{\text{inv}}(f, Q)$ is finite;

ii) $a_\tau(f, Q)$ is finite for some τ;

iii) $a_\tau(f, Q)$ is finite for all τ.

Example 2.1.17. *Consider the following control system in $\mathbb{R}^2 \backslash \{(0,0)\}$*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \left(\frac{x(t)}{\sqrt{x(t)^2 + y(t)^2}} - \omega(t) \right)^2 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where $\omega(t) \in [-1,1]$, $Q := \{(x,y) \in \mathbb{R}^2;\ \frac{1}{2} \leq \sqrt{x^2+y^2} \leq 1\}$ and $K := \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$, then (K,Q) is an admissible pair. In fact, given any $z = (x,y) \in K$, you can consider the control $\omega_z(t) :\equiv x$ and note that z can be kept in Q for any positive time $\tau > 0$ by using the constant control function ω_z. Hence $h_{\mathrm{inv}}(K,Q) = \infty$ and by Remark 2.1.12, $P_{\mathrm{inv}}(f,K,Q) = \infty$, for each $f \in C(U,\mathbb{R})$. This example shows the necessity of openess of Q in Proposition 2.1.4: since one needs infinitely many of these control functions for all points on K, given a (τ,K,Q)-spanning $\mathcal{S}$, none finite (τ,K,Q)-spanning set $\mathcal{S}' \subset \mathcal{S}$ can exists.

We can not replace the limit superior in Definition 2.1.1 by a limit, because this limit does not exist in the general case. However, if Q is compact controlled invariant and $K = Q$ we can do that.

Proposition 2.1.18. *If Q is a compact controlled invariant set and $f \in C(U,\mathbb{R})$, then the function $\tau \mapsto a_\tau(f,Q)$ is subadditive and therefore*

$$P_{\mathrm{inv}}(f,Q) = \lim_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f,Q) = \inf_{\tau>0} \frac{1}{\tau} \log a_\tau(f,Q).$$

Proof. If $a_\tau(f,Q) = \infty$ for all τ, the assertion is trivial. Hence, by Corollary 2.1.14 (ii) we can assume that $a_\tau(f,Q) < \infty$ for all τ. If we show that $a_{\tau_1+\tau_2}(f,Q) \leq a_{\tau_1}(f,Q)a_{\tau_2}(f,Q)$ for all $\tau_1,\tau_2 > 0$, then the result follows from Lemma 1.4.9 . To this end, consider for $j = 1,2$ (τ_j,Q)-spanning sets $\mathcal{S}_j$. For $\omega_1 \in \mathcal{S}_1, \omega_2 \in \mathcal{S}_2$ define a control function $\omega \in \mathcal{U}$ by

$$\omega(t) = \begin{cases} \omega_1(t), & \text{if} \quad t \in [0,\tau_1] \\ \omega_2(t-\tau_1), & \text{if} \quad t > \tau_1 \end{cases}.$$

These functions form a $(\tau_1+\tau_2,Q)$-spanning set. Hence $a_{\tau_1+\tau_2}(f,Q) \leq a_{\tau_1}(f,Q)a_{\tau_2}(f,Q)$, which concludes the proof. □

2.2 Elementary Properties

This section brings together many interesting properties of invariance pressure which allows us to compare with the well-known properties satisfied by the topological pressure of dynamical systems (cf. [39, Section 9.2]). Moreover, all the elementary properties which invariance entropy satisfies (cf. [27, Section 2.2]) can be generalized when we assign a weigh f on the control values.

We start this section with the following theorem which shows that for the invariance pressure the time may be discretized, which generalizes the Proposition 3.4 (ii) of [9].

Theorem 2.2.1. *If (K, Q) is an admissible pair for $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ and $f \in C(U, \mathbb{R})$, then for each $\tau > 0$*

$$P_{\text{inv}}(f, K, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f, K, Q). \tag{2.2-4}$$

Proof. Given $f \in C(U, \mathbb{R})$, the inequality

$$P_{\text{inv}}(f, K, Q) \geq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f, K, Q)$$

is obvious. For the converse, we can see that the function $g(u) := f(u) - \inf f$ is non-negative (if $f \geq 0$, it is not necessary to consider the function g). Let $(\tau_k)_{k\geq 1}$, $\tau_k \in (0, \infty)$ and $\tau_k \to \infty$. Then for every $k \geq 1$ there exists $n_k \geq 1$ such that $n_k\tau \leq \tau_k \leq (n_k + 1)\tau$ and $n_k \to \infty$ for $k \to \infty$. Since $g \geq 0$ it follows that

$$a_{\tau_k}(g, K, Q) \leq a_{(n_k+1)\tau}(g, K, Q)$$

and consequently

$$\frac{1}{\tau_k} \log a_{\tau_k}(g, K, Q) \leq \frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(g, K, Q).$$

This yields

$$\limsup_{k\to\infty} \frac{1}{\tau_k} \log a_{\tau_k}(g, K, Q) \leq \limsup_{k\to\infty} \frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(g, K, Q).$$

Since

$$\frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(g, K, Q) = \frac{n_k + 1}{n_k} \frac{1}{(n_k + 1)\tau} \log a_{(n_k+1)\tau}(g, K, Q)$$

and $\frac{n_k+1}{n_k} \to 1$ for $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \frac{1}{\tau_k} \log a_{\tau_k}(g, K, Q) \leq \limsup_{k\to\infty} \frac{1}{n_k\tau} \log a_{n_k\tau}(g, K, Q) \leq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(g, K, Q).$$

This shows that

$$P_{\text{inv}}(f - \inf f, K, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f - \inf f, K, Q),$$

and as in Proposition 2.1.11 (ii) we have

$$\begin{aligned} P_{\text{inv}}(f, K, Q) &= P_{\text{inv}}(f - \inf f, Q) + \inf f = P_{\text{inv}}(g, K, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f - \inf f, K, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log e^{-n \inf f} a_{n\tau}(f, K, Q) + \inf f \\ &= \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f, K, Q). \end{aligned}$$

□

In the case of dynamical systems, the topological pressure tends to be smaller when we restrict the system to invariant subsets than when the whole space is considered [39, cf. Theorem 9.8 (iii)]. Now, given an admissible pair (K, Q) we investigate how the invariance pressure behaves when we consider a finite subcover of K by compact sets. We will need the following lemma which is proved in [27, Lemma 2.1].

Lemma 2.2.2. *For any functions $f_1, \cdots, f_N : \mathbb{T} \cap (0, \infty) \to \mathbb{R}$, $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$, it holds that*

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log \sum_{i=1}^{N} f_i(\tau) \leq \max_{1\leq i\leq N} \limsup_{\tau\to\infty} \frac{1}{\tau} \log f_i(\tau).$$

Proposition 2.2.3. *Let $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ be a control system, $f \in C(U, \mathbb{R})$ and (K, Q) an admissible pair. Assume that $K = \bigcup_{i=1}^{N} K_i$ with finitely many compact sets $K_1, \cdots, K_N$. Then each pair (K_i, Q), $i \in \{1, \cdots, N\}$ is admissible and*

$$P_{\text{inv}}(f, K, Q) = \max_{1\leq i\leq N} P_{\text{inv}}(f, K_i, Q).$$

Proof. By Proposition 2.1.7 (iii), we have that each (K_i, Q) is admissible and $P_{\text{inv}}(f, K_i, Q) \leq P_{\text{inv}}(f, K, Q)$, for all $i \in \{1, \cdots, N\}$, hence

$$\max_{1\leq i\leq N} P_{\text{inv}}(f, K_i, Q) \leq P_{\text{inv}}(f, K, Q).$$

In order to show the reverse inequality, given $\varepsilon > 0$ and $i \in \{1, \cdots, N\}$, there is (τ, K_i, Q)-spanning set $\mathcal{S}_i$ such that

$$a_\tau(f, K_i, Q) \leq \sum_{\omega \in \mathcal{S}_i} e^{(S_\tau f)(\omega)} < a_\tau(f, K_i, Q) + \frac{\varepsilon}{N}.$$

Then $\mathcal{S} := \bigcup_{i=1}^N \mathcal{S}_i$ is a (τ, K, Q)-spanning and

$$a_\tau(f, K, Q) \leq \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \leq \sum_{i=1}^N \sum_{\omega \in \mathcal{S}_i} e^{(S_\tau f)(\omega)} < \sum_{i=1}^N a_\tau(f, K_i, Q) + \varepsilon$$

which shows that

$$a_\tau(f, K, Q) \leq \sum_{i=1}^N a_\tau(f, K_i, Q).$$

By Lemma 2.2.2 we obtain

$$P_{\text{inv}}(f, K, Q) \leq \limsup_{\tau \to \infty} \frac{1}{\tau} \log \sum_{i=1}^N a_\tau(f, K_i, Q) \leq \max_{1 \leq i \leq N} P_{\text{inv}}(f, K_i, Q).$$

□

Remark 2.2.4. *Note that Proposition 2.2.3 can not be generalized to the case of a countable cover of K. For instance, see [27, Remark 7.3] when f is the null function.*

Next we discuss changes in the considered set Q.

Proposition 2.2.5. *Let $f \in C(U, \mathbb{R})$ and $Q \subset X$ a compact controlled invariant set. Assume that $Q = \bigcup_{i=1}^N Q_i$ with compact controlled invariant sets $Q_1, \ldots, Q_N$. Then*

$$P_{\text{inv}}(f, Q) \leq \max_{1 \leq i \leq N} P_{\text{inv}}(f, Q_i).$$

Proof. For every $i \in \{1, \ldots, N\}$, let $\mathcal{S}_i$ a (τ, Q_i)-spanning set and define $\mathcal{S} := \bigcup_{i=1}^N \mathcal{S}_i$. Then $\mathcal{S}$ is a (τ, Q)-spanning set with

$$\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \leq \sum_{i=1}^N \sum_{\omega \in \mathcal{S}_i} e^{(S_\tau f)(\omega)}.$$

With

$$a_\tau(f, Q_i) = \inf \left\{ \sum_{\omega \in \mathcal{S}_i} e^{(S_\tau f)(\omega)};\ \mathcal{S}_i\ (\tau, Q_i)\text{-spanning} \right\},$$

we have $a_\tau(f,Q) \leq \sum_{i=1}^N a_\tau(f,Q_i)$. Now Lemma 2.2.2 implies that

$$P_{\mathrm{inv}}(f,Q) \leq \limsup_{\tau\to\infty} \frac{1}{\tau} \log \sum_{i=1}^N a_\tau(f,Q_i) \leq \max_{1\leq i\leq N} P_{\mathrm{inv}}(f,Q_i).$$

□

The next result concerns to the invariance pressure on the product of two control systems. To this end, consider $\Sigma_1 = (\mathbb{R}, M_1, U_1, \mathcal{U}_1, \varphi_1)$ and $\Sigma_2 = (\mathbb{R}, M_2, U_2, \mathcal{U}_2, \varphi_2)$. Then we can built the control system $\Sigma_p = (\mathbb{R}, M_1 \times M_2, U_1 \times U_2, \mathcal{U}_1 \times \mathcal{U}_2, \varphi_1 \times \varphi_2)$ where $\varphi_1 \times \varphi_2 : \mathbb{R} \times (M_1 \times M_2) \times (\mathcal{U}_1 \times \mathcal{U}_2)$ is given by

$$(\varphi_1 \times \varphi_2)(\tau, z, \omega) = (\varphi_1 \times \varphi_2)(\tau, (x,y), (\omega_1, \omega_2)) = (\varphi_1(\tau, x, \omega_1), \varphi_2(\tau, y, \omega_2)).$$

Proposition 2.2.6. *Let $f_i \in C(U_i, \mathbb{R})$ and let (K_i, Q_i) be an admissible pair for Σ_i, $i = 1, 2$. Then*

$$P_{\mathrm{inv}}(f_1 \times f_2, K_1 \times K_2, Q_1 \times Q_2; \Sigma_p) \leq P_{\mathrm{inv}}(f_1, K_1, Q_1; \Sigma_1) + P_{\mathrm{inv}}(f_2, K_2, Q_2; \Sigma_2),$$

where $f_1 \times f_2 \in C(U_1 \times U_2, \mathbb{R})$ is defined by $(f_1 \times f_2)(u,v) = f_1(u) + f_2(v)$.

Proof. Note that $(K_1 \times K_2, Q_1 \times Q_2)$ is admissible for Σ_p. Furthermore, for each $\varepsilon > 0$ there is a (τ, K_i, Q_i)-spanning set $\mathcal{S}_i$, $i = 1, 2$, such that

$$0 < a_\tau(f_i, K_i, Q_i) \leq \sum_{\omega_i \in \mathcal{S}_i} e^{(S_\tau f_i)(\omega_i)} < a_\tau(f_i, K_i, Q_i) + \delta, \;\; i = 1, 2,$$

where

$$\delta := \frac{\sqrt{[a_\tau(f_1, K_1, Q_1) + a_\tau(f_2, K_2, Q_2)]^2 + 4\varepsilon} - [a_\tau(f_1, K_1, Q_1) + a_\tau(f_2, K_2, Q_2)]}{2}.$$

Then $\mathcal{S} := \mathcal{S}_1 \times \mathcal{S}_2 \subset \mathcal{U}_1 \times \mathcal{U}_2$ is a $(\tau, K_1 \times K_2, Q_1 \times Q_2)$-spanning set and

$$\begin{aligned} a_\tau(f_1 \times f_2, K_1 \times K_2, Q_1 \times Q_2) &\leq \sum_{\omega \in \mathcal{S}} e^{(S_\tau (f_1 \times f_2))(\omega)} = \sum_{(\omega_1,\omega_2) \in \mathcal{S}_1 \times \mathcal{S}_2} e^{(S_\tau f_1))(\omega_1)} e^{(S_\tau f_2))(\omega_2)} \\ &= \sum_{\omega_1 \in \mathcal{S}_1} e^{(S_\tau f_1))(\omega_1)} \sum_{\omega_2 \in \mathcal{S}_2} e^{(S_\tau f_2))(\omega_2)} \\ &< a_\tau(f_1, K_1, Q_1) \cdot a_\tau(f_2, K_2, Q_2) + \varepsilon. \end{aligned}$$

Taking ε decreasing to 0 we get

$$a_\tau(f_1 \times f_2, K_1 \times K_2, Q_1 \times Q_2) \le a_\tau(f_1, K_1, Q_1) \cdot a_\tau(f_2, K_2, Q_2).$$

Therefore

$$\begin{aligned} P_{\text{inv}}(f_1 \times f_2, K_1 \times K_2, Q_1 \times Q_2) &= \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_n(f_1 \times f_2, K_1 \times K_2, Q_1 \times Q_2) \\ &\le \limsup_{\tau\to\infty} \frac{1}{\tau} \log\left[a_\tau(f_1, K_1, Q_1) \cdot a_\tau(f_2, K_2, Q_2)\right] \\ &= P_{\text{inv}}(f_1, K_1, Q_1) + P_{\text{inv}}(f_2, K_2, Q_2). \end{aligned}$$

□

Open Question 3. *The topological pressure of a continuous transformation on a compact metric space satisfies the equality presented in Proposition 2.2.6 (see [39, Theorem 9.8 (iv)]). Is this equality verified in the case of invariance pressure, even in the case when $K = Q$?*

Next we prove the power rule for invariance pressure. Consider initially a continuous-time control system $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ and an admissible pair (K, Q). For every real number $s > 0$, we can define $\Sigma_s = (\mathbb{R}, M, U, \mathcal{U}, \varphi^s)$ given by the differential equations

$$\dot{x}(t) = s \cdot F(x(t), \omega(t)),\ \omega \in \mathcal{U},$$

whose trajectories are given by $\varphi^s(t, x, \omega) = \varphi(st, x, \tilde{\omega})$, where $\tilde{\omega}(t) := \omega(t/s)$. In fact,

$$\begin{aligned} \frac{d}{dt}\varphi^s(t, x, \omega) &= \frac{d}{dt}\varphi(st, x, \tilde{\omega}) = s \cdot F(\varphi(st, x, \tilde{\omega}), \tilde{\omega}(st)) \\ &= s \cdot F(\varphi(st, x, \tilde{\omega}), \omega(t)) = s \cdot F(\varphi^s(t, x, \omega), \omega(t)) \end{aligned}$$

and the result follows from uniqueness of solution. The Power rule is stated as:

Proposition 2.2.7. *If $s > 0$ and (K, Q) is an admissible pair for Σ, then it is also admissible for Σ_s and for each $f \in C(U, \mathbb{R})$*

$$P_{\text{inv}}(f, K, Q; \Sigma_s) = s \cdot P_{\text{inv}}(f, K, Q; \Sigma).$$

Proof. It is easy to see that (K, Q) is admissible for Σ_s. Note that if $\mathcal{S} \subset \mathcal{U}$ is an $(s\tau, K, Q)$-spanning set for Σ, then $\mathcal{S}_s := \{\omega(s + \cdot);\ \omega \in \mathcal{S}\}$ is a (τ, K, Q)-spanning set for Σ_s with the same number of elements. Analogously, every (τ, K, Q)-spanning set for Σ_s gives an $(s\tau, K, Q)$-spanning set for Σ with the same number of elements. This proves that

$$a_{s\tau}(f, K, Q; \Sigma) = a_\tau(f, K, Q; \Sigma_s), \ \ \forall\, \tau > 0.$$

Hence, by Proposition 2.2.1 we get

$$\begin{aligned} P_{\text{inv}}(f, K, Q; \Sigma_s) &= \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f, K, Q; \Sigma_s) = \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_{s\tau}(f, K, Q; \Sigma) \\ &= s \cdot \limsup_{\tau\to\infty} \frac{1}{s\tau} \log a_{s\tau}(f, K, Q; \Sigma) = s \cdot P_{\text{inv}}(f, K, Q; \Sigma). \end{aligned}$$

□

2.2.1 Invariance pressure under conjugacy

We reserve this subsection to show that the invariance pressure is preserved by a certain kind of conjugacy between two control systems, which we define in sequence. The invariance under conjugacy is also satisfied for topological pressure, in the case of dynamical systems, and for invariance entropy, in the case of control systems. However the conditions of conjugacy which we require here is stronger than those required in [27, Definition 2.4].

Definition 2.2.8. *Consider $\Sigma_1 = (\mathbb{R}, M_1, U_1, \mathcal{U}_1, \varphi_1)$ and $\Sigma_2 = (\mathbb{R}, M_2, U_2, \mathcal{U}_2, \varphi_2)$ two control systems. Let $\pi : \mathbb{R}_+ \times M_1 \to M_2$, $(t, x) \mapsto \pi_t(x)$, and $H : U_1 \to U_2$ be continuous maps such that the induced map $h_H : \mathcal{U}_1 \to \mathcal{U}_2$, $h_H(\omega)(t) := H(\omega(t))$ for all $t \in \mathbb{R}$, satisfies*

$$\pi_t(\varphi_1(t, x, \omega)) = \varphi_2(t, \pi_0(x), h_H(\omega)) \text{ for all } t \in \mathbb{R}_+, x \in M_1 \text{ and } \omega \in \mathcal{U}_1.$$

Then (π, H) is called a **time-variant semi-conjugacy** *from Σ_1 to Σ_2. If the maps $\pi_t : M_1 \to M_2$, $t \in \mathbb{R}_+$, and $H : U_1 \to U_2$ are homeomorphisms, we call (π, H) a* **time-variant conjugacy** *from Σ_1 to Σ_2.*

Analogously we define a **time-invariant semi-conjugacy** *and* **conjugacy** *from Σ_1 to Σ_2 if π is independent of $t \in \mathbb{R}_+$.*

Proposition 2.2.9. *Consider two control systems as in Definition 2.2.8 and let (π, H) be a time-variant semi-conjugacy from Σ_1 to Σ_2. Further assume that (K, Q) is an admissible pair* *for Σ_1 and*

$$\pi_t(Q) \subset \pi_0(Q) \text{ for all } t > 0.$$

Then $(\pi_0(K), \pi_0(Q))$ is an admissible pair for Σ_2 and

$$P_{\text{inv}}(f \circ H, K, Q; \Sigma_1) \geq P_{\text{inv}}(f, \pi_0(K), \pi_0(Q); \Sigma_2)$$

for all $f \in C(U_2, \mathbb{R})$. Moreover, if Q is compact and the family $\{\pi_t\}_{t \in \mathbb{R}_+}$ is pointwise equicontinuous, then

$$P_{\text{inv,out}}(f \circ H, K, Q; \Sigma_1) \geq P_{\text{inv,out}}(f, \pi_0(K), \pi_0(Q); \Sigma_2)$$

for all $f \in C(U_2, \mathbb{R})$.

Proof. In order to show that $(\pi_0(K), \pi_0(Q))$ is an admissible pair, note that since π is continuous, the set $\pi_0(K)$ is compact. Let $y \in \pi_0(K)$, then $y = \pi_0(x)$ for some $x \in K$. Since (K, Q) is an admissible pair, there is $\omega \in \mathcal{U}_1$ such that $\varphi(\mathbb{R}_+, x, \omega) \subset Q$, and we obtain

$$\varphi_2(t, y, h_H(\omega)) = \pi_t(\varphi_1(t, x, \omega)) \in \pi_t(Q) \subset \pi_0(Q).$$

Therefore $(\pi_0(K), \pi_0(Q))$ is an admissible pair for Σ_2.

Now, let $\mathcal{S} \subset \mathcal{U}_1$ be a (τ, K, Q)-spanning set. With the same arguments as above, we find that $h_H(\mathcal{S}) \subset \mathcal{U}_2$ is $(\tau, \pi_0(K), \pi_0(Q))$-spanning. Hence

$$\sum_{\mu \in h_H(\mathcal{S})} e^{(S_\tau f)(\mu)} = \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(H \circ \omega)} = \sum_{\omega \in \mathcal{S}} e^{(S_\tau (f \circ H))(\omega)}$$

for every (τ, K, Q)-spanning set $\mathcal{S}$, which implies that

$$a_\tau(f, \pi_0(K), \pi_0(Q)) \leq a_\tau(f \circ H, K, Q).$$

Therefore $P_{\text{inv}}(f, \pi_0(K), \pi_0(Q)) \leq P_{\text{inv}}(f \circ H, K, Q)$.

Now assume that Q is compact. Let ϱ_1 denote a metric on M_1 and ϱ_2 a metric on M_2. By compactness of Q, the pointwise equicontinuity of $\{\pi_t\}_{t \in \mathbb{R}_+}$ on Q is uniform, hence for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}_+$, $x \in Q$ and $y \in M_1$ the condition $\varrho_1(x, y) < \delta$ implies $\varrho_2(\pi_t(x), \pi_t(y)) < \varepsilon$.

Let $\mathcal{S} \subset \mathcal{U}_1$ be a $(\tau, K, N_\delta(Q))$-spanning set with $\delta = \delta(\varepsilon)$ as above. Note that if

$y \in \pi_0(K)$, then $y = \pi_0(x)$ for some $x \in K$. For $\omega \in \mathcal{S}$ such that $\varphi_1([0,\tau], x, \omega) \subset N_\delta(Q)$ and for each $t \in [0,\tau]$, there exists $x_t \in Q$ with $\varrho_1(x_t, \varphi_1(t,x,\omega)) < \delta$. This implies that for all $t \in [0,\tau]$

$$\varrho_2(\varphi_2(t,y,h_H(\omega)), \pi_t(x_t)) = \varrho_2(\pi_t(\varphi_1(t,x,\omega)), \pi_t(x_t)) < \varepsilon.$$

This shows that $h_H(\mathcal{S}) \subset \mathcal{U}_2$ is a $(\tau, \pi_0(K), N_\varepsilon(\pi_0(Q)))$-spanning set. We conclude that $a_\tau(f, \pi_0(K), N_\varepsilon(\pi_0(Q))) \leq a_\tau(f \circ H, K, N_\varepsilon(Q))$, and hence

$$P_{\text{inv,out}}(f, \pi_0(K), \pi_0(Q)) \leq P_{\text{inv,out}}(f \circ H, K, Q).$$

□

Remark 2.2.10. *It is easy to see that if (π, H) is a time-variant conjugacy from Σ_1 to Σ_2, then (ψ, H^{-1}) with $\psi_t(y) := \pi_t^{-1}(y)$ is a time-variant conjugacy from Σ_2 to Σ_1. In this case, we have, under the assumptions of the previous proposition,*

$$P_{\text{inv}}(f \circ H, K, Q; \Sigma_1) = P_{\text{inv}}(f, \pi_0(K), \pi_0(Q)); \Sigma_2).$$

A similar argument holds for time-invariant conjugacies.

Example 2.2.11. *Consider two linear control systems in $\mathbb{R}^d$*

$$\Sigma_1 : \ \dot{x}(t) = A_1 x(t) + B_1 \omega(t) \text{ and } \ \Sigma_2 : \ \dot{x}(t) = A_2 x(t) + B_2 \omega(t),$$

where $\omega(t)$ is in a compact set $U \subset \mathbb{R}^m$ for all $t \in \mathbb{R}$, $A_i \in \mathbb{R}^{d\times d}$ and $B_i \in \mathbb{R}^{d\times m}$ for $i = 1, 2$. If there is a nonsingular $d \times d$ matrix T such that $A_2 = TA_1T^{-1}$ and $B_2 = TB_1$, then (T, id_U) is a time-invariant conjugacy from Σ_1 to Σ_2. In fact

$$\begin{aligned}
T(\varphi_1(t,x,\omega)) &= T\left(e^{tA_1}x + \int_0^t e^{(t-s)A_1} B_1\omega(s)ds\right) \\
&= T\left(e^{tT^{-1}A_2T}x + \int_0^t e^{(t-s)T^{-1}A_2T} T^{-1}B_2\omega(s)ds\right) \\
&= T\left(T^{-1}e^{tA_2}Tx + \int_0^t T^{-1}e^{(t-s)A_2}TT^{-1}B_2\omega(s)ds\right) \\
&= e^{tA_2}Tx + \int_0^t e^{(t-s)A_2}B_2\omega(s)ds = \varphi_2(t, Tx, h_{id_U}(\omega)).
\end{aligned}$$

In this case, it follows for every admissible pair (K,Q)*, and all* $f \in C(U,\mathbb{R})$

$$P_{\mathrm{inv}}(f,K,Q;\Sigma_1) = P_{\mathrm{inv}}(f,T(K),T(Q));\Sigma_2).$$

Remark 2.2.12. *All the results of this section can be made for discrete-time control systems, assuming that the control-valued space* U *is compact.*

Remark 2.2.13. *At the beginning of the twentieth century, Caratheodory and Hausdorff originated the notion of dimension of invariant sets, one of the most important characteristics of dynamical systems. The fact that the topological pressure is a characteristic of dimension type was first noticed in [33] and (implicitly) by Bowen [5]. The dimensional approach of invariance entropy was introduced recently by Huang and Zhong [23] and the dimension characteristics of invariance pressure was presented also by Zhong and Huang [42].*

CHAPTER 3

INVARIANCE PRESSURE ON SPECIAL SETS

In this chapter we present other ways to derive the invariance pressure of a continuous-time control system $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ when the set Q of an admissible pair (K, Q) satisfies particular properties. The arguments used here is an adaptation of the results presented in Kawan [26] and [27, Sections 2.2 and 4.2]. Firstly, when Q is an isolated set, that is, a set where the trajectories that are reasonably close Q are actually inside Q. In this case, we will see that the limit for $\varepsilon \searrow 0$ in the definition of $P_{\text{inv,out}}(f, K, Q)$ becomes superfluous. Secondly, for a inner control set Q, or in other words, a set where the control system is controllable (in a sense that we will formulate more rigorously) in a neighborhood of it and, in this situation, we will obtain that the limit superior in this definition can be replaced by a limit inferior.

Thirdly, we investigate how the invariance pressure of a pair (K, Q) behaves when Q is a control set (or a closure of it) with nonempty interior, and we will note that the invariance pressure is constant when we vary the compacts with nonempty interior K inside Q. Adapting the ideas from [25, Theorem 4.4], we also get an upper bound for the invariance pressure of an admissible pair (K, Q) of the linear control system in terms of the spectrum of the matrix A. In the last section we get lower bounds for the invariance pressure in terms of volume growth rates for a control system on a Riemannian manifold.

3.1 Isolated Sets

In the dynamical systems environment, the definitions of topological pressure via separated and spanning sets of an expansive homeomorphism are not necessary to take the limit $\varepsilon \searrow 0$ as we can see in [39, Theorem 9.6 (ii)]. This consequence also holds for invariance pressure when we are considering isolated sets presented in [24].

We assume that Σ satisfies the following additional properties:

1) The set $\mathcal{U}$ of admissible control functions is endowed with a topology that makes it a sequentially compact space, that is, every sequence in $\mathcal{U}$ has a convergent subsequence;

2) The solution map $\varphi : \mathbb{R}_+ \times M \times \mathcal{U} \to M$ is continuous when $\mathcal{U}$ is endowed with the above topology.

These properties are satisfied in particular for a control-affine system (see Example 1.3.2) when $\mathcal{U}$ is provided with the weak* topology.

Remember that for $x \in M$ and $A \subset M$, the distance between x to A, denoted by dist(x, A), is defined by

$$\text{dist}(x, A) = \inf_{y \in A} d(x, y),$$

where d is a metric on M.

A compact set $Q \subset M$ is called **isolated** if there exists $\delta_0 > 0$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ the following implication holds:

$$\varphi(\mathbb{R}_+, x, \omega) \subset \overline{N_{\delta_0}(Q)} \Rightarrow \varphi(\mathbb{R}_+, x, \omega) \subset Q. \tag{3.1-1}$$

The lemma in sequence will help us to show the main result of this section. The proof can be found in [27, Lemma A.3]

Lemma 3.1.1. *Let (X, ϱ) be a locally compact metric space. Then for every compact set $K \subset X$ there exists some $\varepsilon > 0$ such that $\overline{N_\varepsilon(K)}$ is compact.*

Proposition 3.1.2. *Let (K, Q) be an admissible pair such that Q is compact and isolated with constant δ_0. Then it holds, for all $f \in C(U, \mathbb{R})$*

$$P_{\text{inv,out}}(f, K, Q) = P_{\text{inv}}(f, K, N_\varepsilon(Q)) \textit{ for all } \varepsilon \in (0, \delta_0],$$

Proof. Since M is locally compact, by Lemma 3.1.1 we may assume that δ_0 is small enough that $\overline{N_{\delta_0}(Q)}$ is compact, since assumption (3.1-1) is also satisfied for smaller δ_0.

By an argument, similar to [27, Proposition 2.2.17], we can see that for all $\rho > 0$ and for all $\varepsilon \in (0, \delta_0]$ there is $n \in \mathbb{N}$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ we get

$$\max_{t \in [0,n]} \text{dist}(\varphi(t, x, \omega), Q) \leq \varepsilon$$

implies dist$(x, Q) < \rho$.

Now let $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_0$. Then there exists $n \in \mathbb{N}$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ it holds that $\max_{t \in [0,n]} \text{dist}(\varphi(t, x, \omega), Q) \leq \varepsilon_2$ implies dist$(x, Q) < \varepsilon_1$. For arbitrary $\tau > 0$, let $\mathcal{S}$ be a $(n+\tau, K, N_{\varepsilon_2}(Q))$-spanning set. For $x \in K$, there exists $\omega_x \in \mathcal{S}$ with $\varphi([0, n + \tau], x, \omega_x) \subset N_{\varepsilon_2}(Q)$. For every $s \in [0, \tau]$, we obtain

$$\max_{t \in [0,n]} \text{dist}(\varphi(t, \varphi(s, x, \omega_x), \Theta_s \omega_x), Q) = \max_{t \in [0,n]} \text{dist}(\varphi(t + s, x, \omega_x), Q) < \varepsilon_2.$$

Hence we have dist$(\varphi(s, x, \omega_x), Q) < \varepsilon_1$ for all $s \in [0, \tau]$, which implies that $\mathcal{S}$ is a $(\tau, K, N_{\varepsilon_1}(Q))$-spanning set. Therefore, given $g \in C(U, \mathbb{R})$, $g \geq 0$, we get

$$a_\tau(g, K, N_{\varepsilon_1}(Q)) \leq a_{n+\tau}(g, K, N_{\varepsilon_2}(Q)), \ \forall \tau > 0,$$

which implies $P_{\text{inv}}(f, K, N_{\varepsilon_1}(Q)) \leq P_{\text{inv}}(f, K, N_{\varepsilon_2}(Q))$, for all $f \in C(U, \mathbb{R})$.

By Proposition 2.1.7 (ii) we have $P_{\text{inv}}(f, K, N_{\varepsilon_2}(Q)) \leq P_{\text{inv}}(f, K, N_{\varepsilon_1}(Q))$ and the proof is complete. □

3.2 Inner Control Sets

In this section, we will show that the limit superior in the definition of invariance pressure of a control set can be replaced by the limit inferior, if certain controllability properties near the control set are satisfied. This change holds in general (and, in fact, is a well-known result) for the case of topological pressure of a dynamical system (see, for instance, [39, Theorem 9.4 (vii) and (viii)]) both in definition via spanning and separated sets. However, for invariance pressure this property is not clear in general.

A control set $D \subset M$ is called an **inner control set** if there exists an increasing family of compact and convex sets $\{U_\rho\}_{\rho\in[0,1]}$ in $\mathbb{R}^m$ (i.e., $U_{\rho_1} \subset U_{\rho_2}$ for $\rho_1 < \rho_2$), such that for every $\rho \in [0,1]$ the control system Σ with control range U_ρ (instead of U) has a control set D_ρ with nonvoid interior and compact closure, and the following conditions are satisfied:

i) $U = U_0$ and $D = D_0$;

ii) $\overline{D_{\rho_1}} \subset \operatorname{int}(D_{\rho_2})$ whenever $\rho_1 < \rho_2$;

iii) For every neighborhood W of $\overline{D}$ there is $\rho \in [0,1)$ with $\overline{D_\rho} \subset W$.

This notion (slightly modified) is taken from Kawan [27, Definition 2.6]. Below, we will consider an inner control set $D = D_0$ (corresponding to the control range $U = U_0$) and characterize the invariance pressure of the controlled invariant set $Q = \overline{D}$ with respect to the larger control range $U_1 \supset U_0$.

The following result shows that for admissible pairs (K, Q) where Q is the closure of an inner control set, the limit superior in the definition of outer invariance pressure can be replaced by the limit inferior. The proof follows [27, Proposition 2.16].

Proposition 3.2.1. *Let Q be the closure of an inner control set D of a control system Σ. Then for every compact set $K \subset D$, the pair (K, Q) is admissible for the control system with control range U_1 and if $\operatorname{int}K \neq \emptyset$ we have*

$$P_{\text{inv,out}}(f, K, Q) = \lim_{\varepsilon\searrow 0} \liminf_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)),\ \forall f \in C(U, \mathbb{R}).$$

Proof. First observe that (by the Tietze extension theorem) every continuous function $f \in C(U, \mathbb{R})$ can be extended to a continuous function $f \in C(U_1, \mathbb{R})$. We fix such an extension. Our proof will show that $P_{\text{inv,out}}(f, K, Q)$ does not depend on this extension.

From conditions (ii) and (iii) of inner control sets, it follows that exists a monotonically decreasing sequence $(\rho_n)_{n\in\mathbb{N}}$ in $[0,1)$ with $D_{\rho_n} \subset N_{1/n}(Q)$ for all $n \in \mathbb{N}$. Since $Q = \overline{D} \subset \operatorname{int}(D_{\rho_n})$ for all $n \in \mathbb{N}$, we can find a monotonically decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers with $\varepsilon_n \searrow 0$ such that $\overline{N_{\varepsilon_n}(Q)} \subset D_{\rho_n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ it is possible to steer all points of $N_{\varepsilon_n}(Q)$ to K with finitely many control functions using the control range U_{ρ_n}. In fact, since $\overline{N_{\varepsilon_n}(Q)}$ and K are subsets of the control set D_{ρ_n} for each n, then for all $x \in \overline{N_{\varepsilon_n}(Q)}$, there exist $t_x^n > 0$ and $\mu_x^n \in \mathcal{U}$,

$\mu_x^n(t) \in U_{\rho_n}$ for all t, such that $\varphi(t_x^n, x, \mu_x^n) \in \mathrm{int}K$ by the approximate controllability of the control set D_{ρ_n}. Continuity implies that there exists a neighborhood W_x^n of x such that $\varphi(t_x^n, W_x^n, \mu_x^n) \subset \mathrm{int}K$. By compactness there exist $x_1^n, \ldots, x_{k_n}^n \in \overline{N_{\varepsilon_n}(Q)}$ such that

$$\overline{N_{\varepsilon_n}(Q)} \subset \bigcup_{i=1}^{k_n} W_{x_i}^n.$$

Denote $\mathcal{S}_n := \{\mu_1^n, \ldots, \mu_{k_n}^n\}$, where $\mu_j^n = \mu_{x_j}^n$, and $\tau_j^n := t_{x_j}^n$. Observe that given $x \in N_{\varepsilon_n}(Q)$, the trajectory $\varphi(t, x, \mu_j^n)$, $t \in [0, \tau_j^n]$, does not leave the control set $D_{\rho_n} \subset N_{1/n}(Q)$ by the no-return property.

For every $\tau > \tau_M^n := \max\{\tau_j^n; j = 1, \ldots k_n\}$ consider a finite $(\tau, K, N_\varepsilon(Q))$-spanning set $\mathcal{S} = \{\omega_1, \ldots, \omega_k\}$, where $\varepsilon \in (0, \varepsilon_n]$ and the controls take values in U_0. Let $\widetilde{\mathcal{S}}$ be the set consisting of the functions

$$\nu_{ij}^n(t) = \begin{cases} \omega_i(t), & \text{if} \quad t \in [0, \tau - \tau_j^n] \\ \mu_j^n(t - \tau_j^n), & \text{if} \quad t \in (\tau - \tau_j^n, \tau] \end{cases}, 1 \le i \le k \text{ and } 1 \le j \le k_n.$$

Thus for every $x \in K$ there is a control in $\tilde{S}$ keeping the corresponding trajectory in $N_\varepsilon(Q)$ up to time $\tau - \tau_j^n$ and then steering the solutions back to K. Now, for $m \in \mathbb{N}$, define $\widehat{\mathcal{S}}$ as the set obtained by m iterations of the elements of $\widetilde{\mathcal{S}}$. Hence $\widehat{\mathcal{S}}$ is a $(m\tau, K, N_{1/n}(Q))$-spanning set with $\#\widehat{\mathcal{S}} \le (\#\mathcal{S})^m \cdot (\#\mathcal{S}_n)^m < \infty$.

We compute for $\nu \in \widehat{\mathcal{S}}$

$$\begin{aligned}
(S_{m\tau} f)(\nu) &= \int_0^{m\tau} f(\nu(t))dt = \int_0^{\tau} f(\nu_{i_1,j_1}(t))dt + \cdots + \int_{(m-1)\tau}^{m\tau} f(\nu_{i_m,j_m}(t))dt \\
&= \int_0^{\tau-\tau_{j_1}} f(\omega_{i_1}(t))dt + \int_{\tau-\tau_{j_1}}^{\tau} f(\mu_{j_1}^n(t - \tau_{j_1}^n))dt + \cdots + \\
&\quad + \int_{(m-1)\tau}^{m\tau-\tau_{j_m}} f(\omega_{i_m}(t - (m-1)\tau))dt + \int_{m\tau-\tau_{j_m}}^{m\tau} f(\mu_{j_m}^n(t - (m\tau - \tau_{j_m}^n)))dt \\
&= \int_0^{\tau-\tau_{j_1}} f(\omega_{i_1}(t))dt + \int_0^{\tau_{j_1}} f(\mu_{j_1}^n(t))dt + \cdots + \\
&\quad + \int_0^{\tau-\tau_{j_m}} f(\omega_{i_m}(t))dt + \int_0^{\tau_{j_m}} f(\mu_{j_m}^n(t))dt \\
&\le (S_\tau f)(\omega_{i_1}) + \cdots + (S_\tau f)(\omega_{i_m}) + 2m\tau_M^n \sup f.
\end{aligned}$$

This implies for all $(\tau, K, N_\varepsilon(Q))$-spanning sets $\mathcal{S}$ and $\varepsilon \in (0, \varepsilon_n]$

$$\begin{aligned}
a_{m\tau}(f, K, N_{1/n}(Q)) &\leq \sum_{\nu \in \widehat{\mathcal{S}}} e^{(S_{m\tau} f)(\nu)} \\
&\leq e^{2m\tau_M^n \sup f} \cdot \sum_{\omega_{i_l} \in \mathcal{S};\ 1 \leq l \leq m} e^{(S_\tau f)(\omega_{i_1}) + \cdots + (S_\tau f)(\omega_{i_m})} \\
&\leq e^{2m\tau_M^n \sup f} \cdot \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right) \cdots \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right) \\
&= e^{2m\tau_M^n \sup f} \cdot \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right)^m .
\end{aligned}$$

It follows that $a_{m\tau}(f, K, N_{1/n}(Q)) \leq e^{2m\tau_M^n \sup f} \cdot (a_\tau(f, K, N_\varepsilon(Q)))^m$ for all $m \in \mathbb{N}$, $\tau > 0$ and $\varepsilon \in (0, \varepsilon_n]$. By discretization of time we get

$$\begin{aligned}
P_{\text{inv}}(f, K, N_{1/n}(Q)) &= \limsup_{m \to \infty} \frac{1}{m\tau} \log a_{m\tau}(f, K, N_{1/n}(Q)) \\
&\leq \limsup_{m \to \infty} \frac{1}{m\tau} (2m\tau_M^n \sup f + m \log a_\tau(f, K, N_\varepsilon(Q))) \\
&= \frac{2}{\tau} \tau_M^n \sup f + \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&P_{\text{inv}}(f, K, N_{1/n}(Q)) \\
&\leq \lim_{\varepsilon \searrow 0} \liminf_{\tau \to \infty} \left(\frac{2}{\tau} \tau_M^n \sup f + \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)) \right) \\
&= \lim_{\varepsilon \searrow 0} \liminf_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)).
\end{aligned}$$

Since this inequality holds for every $n \in \mathbb{N}$, the assertion follows. □

Remark 3.2.2. *Note that it does not necessarily follow that the limit*

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q))$$

exists for any $\varepsilon > 0$.

3.3 Control Sets

In this section we present an interesting fact that invariance pressure of control sets satisfies: If D is a control set with nonempty interior, then for all compact $K \subset D$, (K, D) is an admissible pair and $P_{\text{inv}}(f, K, D)$ does not depend on K, provided that K has nonvoid interior. This fact will be extremely important to obtain an upper bound for $P_{\text{inv}}(f, K, Q)$ of linear control systems in terms of the positive real parts of the eigenvalues of A.

As an application we will consider a simple mechanical system composed by a pendulum where external torques are applied in order to "control" its position and the velocity.

The ideas used in this section follow from Kawan [25] and [27, Chapter 5].

Proposition 3.3.1. *Let $Q \subset M$ be a set with the no-return property. Assume that (K_1, Q) and (K_2, Q) are two admissible pairs such that K_2 has nonempty interior and*

$$\forall x \in K_1 \ \exists \omega_x \in \mathcal{U} \ \exists \tau_x > 0 : \varphi(\tau_x, x, \omega_x) \in \text{int}K_2.$$

Then for all $f \in C(U, \mathbb{R})$

$$P_{\text{inv}}(f, K_1, Q) \leq P_{\text{inv}}(f, K_2, Q).$$

Proof. Note that if there exists τ_0 such that $a_\tau(f, K_2, Q) = +\infty$ for all $\tau \geq \tau_0$, then $P_{\text{inv}}(f, K_2, Q) = +\infty$ and hence the assertion holds.

If this is not the case, we can get a sequence $\tau_k \to \infty$ such that $a_{\tau_k}(f, K_2, Q)$ is finite for all k. For all $x \in K_1$, let $\omega_x \in \mathcal{U}$ and $\tau_x > 0$ as in the assumption. Since $\varphi(\tau_x, \cdot, \omega_x)$ is continuous, we find, for every $x \in K_1$, an open neighborhood V_x of x such that $\varphi(\tau_x, V_x, \omega_x) \subset \text{int}K_2$. By the no-return property of Q, we have $\varphi([0, \tau_x], y, \omega_x) \subset Q$, for all $y \in K_1 \cap V_x$. The family $\{V_x\}_{x \in K_1}$ is an open cover of K_1 and by compactness there exist $x_1, \ldots, x_n \in K_1$ with $K_1 \subset \cup_{i=1}^n V_{x_i}$. Now, let $\mathcal{S} := \{\mu_1, \ldots, \mu_k\}$ be a finite (τ, K_2, Q)-spanning set, for some $\tau > \tau_M - \tau_m$, where $\tau_M := \max_{1 \leq i \leq n} \tau_{x_i}$ and $\tau_m := \min_{1 \leq i \leq n} \tau_{x_i}$.

For every index pair (i, j) with $1 \leq i \leq n$, $1 \leq j \leq k$ such that there exists $x \in K_1$ with $y_x := \varphi(\tau_{x_i}, x, \omega_{x_i}) \in \text{int}K_2$ and $\varphi([0, \tau], y_x, \mu_j) \subset Q$, we can define a control function

$$\nu_{ij}(t) = \begin{cases} \omega_{x_i}(t), & \text{if} \quad t \in [0, \tau_{x_i}] \\ \mu_j(t - \tau_{x_i}), & \text{if} \quad t > \tau_{x_i} \end{cases}.$$

Define the set $\widetilde{\mathcal{S}}$ of all these control functions. Let $\widetilde{\tau} := \tau + \tau_m$, hence $\tau \geq \widetilde{\tau} - \tau_M$. Then $\widetilde{\mathcal{S}}$ is a $(\widetilde{\tau}, K_1, Q)$-spanning set by construction, and consequently, for all $f \in C(U, \mathbb{R})$, $f \geq 0$, we have

$$\begin{aligned}
(S_{\widetilde{\tau}} f)(\nu_{ij}) &= (S_{\tau_{x_i}} f)(\omega_{x_i}) + \int_{\tau_{x_i}}^{\widetilde{\tau}} f(\mu_j(t - \tau_{x_i}))dt \\
&= (S_{\tau_{x_i}} f)(\omega_{x_i}) + \int_{0}^{\widetilde{\tau} - \tau_{x_i}} f(\mu_j(t))dt \\
&= (S_{\tau_{x_i}} f)(\omega_{x_i}) + (S_{\widetilde{\tau} - \tau_{x_i}} f)(\mu_j) \\
&\leq (S_{\tau_{x_i}} f)(\omega_{x_i}) + (S_{\tau} f)(\mu_j).
\end{aligned}$$

Hence

$$\begin{aligned}
a_\tau(f, K_1, Q) \leq a_{\widetilde{\tau}}(f, K_1, Q) &\leq \sum_{\nu_{ij} \in \widetilde{\mathcal{S}}} e^{(S_{\widetilde{\tau}} f)(\nu_{ij})} \leq \sum_{1 \leq i \leq n,\ \mu \in \mathcal{S}} e^{(S_{\tau_{x_i}} f)(\omega_{x_i})} e^{(S_\tau f)(\mu)} \\
&\leq \sum_{1 \leq i \leq n} e^{(S_{\tau_{x_i}} f)(\omega_{x_i})} \cdot \sum_{\mu \in \mathcal{S}} e^{(S_\tau f)(\mu)} \leq n e^{\|f\| \tau_M} \sum_{\mu \in \mathcal{S}} e^{(S_\tau f)(\mu)},
\end{aligned}$$

because $0 \leq \widetilde{\tau} - \tau_{x_i} \leq \tau$. Since this inequality holds for all finite (τ, K_2, Q)-spanning sets, we have

$$a_\tau(f, K_1, Q) \leq n e^{\|f\| \tau_M} a_\tau(f, K_2, Q),\ \tau > \tau_M - \tau_m,$$

where $ne^{\|f\|\tau_M}$ is constant in τ. Therefore, we obtain for all $f \in C(U, \mathbb{R})$,$f \geq 0$,

$$P_{\text{inv}}(f, K_1, Q) \leq P_{\text{inv}}(f, K_2, Q).$$

Now consider an arbitrary $f \in C(U, \mathbb{R})$. Then $\tilde{f} \in C(U, \mathbb{R})$ given by $\tilde{f}(u) = f(u) - \inf f$ satisfies $\tilde{f} \geq 0$. Using Proposition 2.1.11 (ii) it follows that

$$\begin{aligned}
P_{\text{inv}}(f, K_1, Q) &= P_{\text{inv}}(\tilde{f}, K_1, Q) + \inf_{u \in U} f(u) \leq P_{\text{inv}}(\tilde{f}, K_2, Q) + \inf_{u \in U} f(u) \\
&= P_{\text{inv}}(f, K_2, Q) - \inf_{u \in U} f(u) + \inf_{u \in U} f(u) \\
&= P_{\text{inv}}(f, K_2, Q).
\end{aligned}$$

□

As an immediate consequence we can see that for all $f \in C(U,\mathbb{R})$, $P_{\text{inv}}(f,\cdot,D)$ is constant when we vary the compacts with nonvoid interior K inside D.

Corollary 3.3.2. *Let $D \subset M$ be a control set and let $K_1, K_2 \subset D$ be two compact subsets with nonempty interior. Then (K_1,D) and (K_2,D) are admissible pairs and for all $f \in C(U,\mathbb{R})$ we have*

$$P_{\text{inv}}(f,K_1,D) = P_{\text{inv}}(f,K_2,D).$$

Proof. This follows, since control sets with nonvoid interior satisfy the no-return property and (K,D) is admissible for any $K \subset D$, because all control sets are controllable invariant. □

We consider a class of linear control systems presented in Example 1.3.1 where A is hyperbolic (that is, A has no eigenvalues on the imaginary axis). Suppose that the pair (A,B) is controllable (that is, $\text{rank}\big[B\ AB\ \cdots\ A^{d-1}B\big] = d$). Consequently, the control system is locally accessible. For simplicity, we will denote such a system Σ_{lin}.

The following result generalizes and improves [6, Theorem 27] (where the outer invariance pressure was considered). The proof follows Kawan [25, Theorem 4.3], [27, Theorem 5.1], considerably simplified for the linear situation.

Theorem 3.3.3. *Consider a linear control system of the form Σ_{lin} and assume that the pair (A,B) is controllable, that A is hyperbolic and the control range U is a compact neighborhood of the origin in $\mathbb{R}^m$. Let D be the unique control set with nonempty interior and let $f \in C(U,\mathbb{R})$.*

Then for every compact set $K \subset D$ the pair (K,D) is admissible and

$$P_{\text{inv}}(f,K,D) \leq \sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda \text{Re}(\lambda)\} + \inf \frac{1}{T}\int_0^T f(\omega(s))ds,$$

where $\sigma(A)$ denotes the spectrum of A, n_λ is the algebraic multiplicity of $\lambda \in \sigma(A)$ and the infimum is taken over all $T > 0$ and all T-periodic controls $\omega(\cdot)$ with a T-periodic trajectory $x(\cdot)$ in $\text{int}D$ such that $\{\omega(t); t \in [0,T]\}$ is contained in a compact subset of $\text{int}U$.

Proof. We will construct a compact subset $K \subset D$ with nonvoid interior such that

$$P_{\text{inv}}(f,K,D) \leq \sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda \text{Re}(\lambda)\} + \inf \frac{1}{T}\int_0^T f(\omega_0(s))ds.$$

Then the assertion will follow, since every compact subset of D is contained in a compact subset K of D with nonvoid interior and the invariance pressure is independent of the choice of such a set K by Corollary 3.3.2.

For the proof consider a τ_0-periodic control $\omega_0(\cdot)$ with τ_0-periodic trajectory as in the statement of the theorem. We can transform A into real Jordan form R without changing the invariance pressure, cf. Example 2.2.11 and, without loss of generality, we can write

$$x_0 = e^{R\tau_0}x_0 + \int_0^{\tau_0} e^{R(\tau_0 - s)} B\omega_0(s)ds. \tag{3.3-2}$$

Step 1: Choose a basis $\mathcal{B}$ of $\mathbb{R}^d$ adapted to the real Jordan structure of R and let $L_1(R), \ldots, L_r(R)$ be the different Lyapunov spaces of R, that is, the sums of the generalized eigenspaces corresponding to eigenvalues with the same real part ρ_j. Then we have the decomposition

$$\mathbb{R}^d = L_1(R) \oplus \cdots \oplus L_r(R).$$

Let $d_j = \dim L_j(R)$ and denote the restriction of R to $L_j(R)$ by R_j. Now take an inner product on $\mathbb{R}^d$ such that the basis $\mathcal{B}$ is orthonormal with respect to this inner product and let $\|\cdot\|$ denote the induced norm.

Step 2: We fix some constants: Let S_0 be a real number which satisfies

$$S_0 > \sum_{j=1}^{r} \max(0, d_j\rho_j)$$

and choose $\xi = \xi(S_0) > 0$ such that

$$0 < d\xi < S_0 - \sum_{j=1}^{r} \max(0, d_j\rho_j).$$

Let $\delta \in (0, \xi)$ be chosen small enough such that $\rho_j < 0$ implies $\rho_j + \delta < 0$ for all j. It follows that there exists a constant $c = c(\delta) \geq 1$ such that for all j and for all $k \in \mathbb{N}$

$$\left\|e^{tR_j}\right\| \leq ce^{(\rho_j+\delta)t} \text{ for all } t \geq 0.$$

For every $t > 0$ we define positive integers

$$M_j(t) = \begin{cases} \lfloor e^{(\rho_j+\xi)t} \rfloor + 1 & \text{if} \quad \rho_j \geq 0 \\ 1 & \text{if} \quad \rho_j < 0 \end{cases}.$$

Moreover, we define a function $\beta : (0, \infty) \to (0, \infty)$ by

$$\beta(t) = \max_{1 \leq j \leq r} \left[e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \right], t > 0.$$

If $\rho_j < 0$, then $\rho_j + \delta < 0$ and $M_j(t) \equiv 1$. This implies that $e^{(\rho_j+\delta)t}/M_j(t)$ converges to zero for $t \to \infty$. If $\rho_j \geq 0$, we have $M_j(t) \geq e^{(\rho_j+\xi)t}$ and hence

$$e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \leq e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{e^{(\rho_j+\xi)t}} = e^{(\delta-\xi)t} \sqrt{d_j}. \tag{3.3-3}$$

Since $\delta \in (0, \xi)$, we have $\delta - \xi < 0$ and hence the terms above converge to zero for $t \to \infty$. Thus, also $\beta(t) \to 0$ for $t \to \infty$. Since we assume controllability of (A, B) there exists $C > 0$ such that for every $\lambda \in \mathbb{R}^d$ there is a control $\omega \in L^\infty(0, \tau, \mathbb{R}^m)$ with

$$\varphi(\tau_0, \lambda, \omega) = e^{R\tau_0}\lambda + \int_0^{\tau_0} e^{R(\tau_0-s)} B\omega(s) ds = 0 \text{ and } \|\omega\|_\infty \leq C \|\lambda\|. \tag{3.3-4}$$

The inequality follows by the inverse mapping theorem.

For $b_0 > 0$ let $\mathcal{C}$ be the d-dimensional compact cube $\mathcal{C}$ in $\mathbb{R}^d$ centered at the origin with sides of length $2b_0$ parallel to the vectors of the basis $\mathcal{B}$. Choose b_0 small enough such that, with $x_0 := x(0)$

$$K := x_0 + \mathcal{C} \subset D$$

and $\overline{B(\omega_0(t), Cb_0)} \subset U$ for almost all $t \in [0, \tau_0]$. This is possible, since $x_0 \in \text{int}D$ and almost all values $\omega_0(t)$ are in a compact subset of the interior of U.

Step 3. Let $\varepsilon > 0$ and $\tau = k\tau_0$ with $k \in \mathbb{N}$. We may take $k \in \mathbb{N}$ large enough such that

$$\frac{d}{\tau} \log 2 < \varepsilon. \tag{3.3-5}$$

Furthermore, we may choose b_0 small enough such that $Cb_0 < \varepsilon$. Partition $\mathcal{C}$ by dividing each coordinate axis corresponding to a component of the jth Lyapunov space $L_j(R)$ into $M_j(\tau)$ intervals of equal length. The total number of subcuboids in this partition of $\mathcal{C}$ is $\prod_{j=1}^{r} M_j(\tau)^{d_j}$.

Next we will show that it suffices to take $\prod_{j=1}^{r} M_j(\tau)^{d_j}$ control functions to steer the solutions from all states in $x_0 + \mathcal{C}$ back to $x_0 + \mathcal{C}$ in time τ such that the controls are within distance ε to ω_0.

Let λ be the center of a subcuboid. By (3.3-4) there exists $\omega \in L^{\infty}(0, \tau, \mathbb{R}^m)$ such that

$$\varphi(\tau, \lambda, \omega) = 0 \text{ and } \|\omega\|_{\infty} \leq C \|\lambda\| \leq C b_0 < \varepsilon.$$

Hence $\omega(t) \in U$ for a.a. $t \in [0, \tau]$ and, using (3.3-2) and linearity, we find that $x_0 + \lambda$ is steered by $\omega_0 + \omega$ in time $\tau = k\tau_0$ to x_0,

$$\varphi(\tau, x_0 + \lambda, \omega_0 + \omega) = \varphi(\tau, x_0, \omega_0) + \varphi(\tau, \lambda, \omega) = x_0. \tag{3.3-6}$$

Now consider an arbitrary point $x \in \mathcal{C}$. Then it lies in one of the subcuboids and we denote the corresponding center of this subcuboid by λ with associated control ω. We will show that $\omega_0 + \omega$ also steers $x_0 + x$ back to $x_0 + \mathcal{C}$. Observe that

$$\|x - \lambda\| \leq \frac{b_0}{M_j(\tau)} \sqrt{d_j}.$$

This implies that

$$\left\|e^{\tau R} x - e^{\tau R} \lambda\right\| \leq \left\|e^{(k\tau_0 R_j)}\right\| \|x - \lambda\| \leq c e^{(\rho_j + \delta) k\tau_0} \frac{b_0}{M_j(k\tau_0)} \sqrt{d_j} \to 0 \text{ for } k \to \infty,$$

and hence for k large enough $\left\|e^{\tau R} x - e^{\tau R} \lambda\right\| \leq b_0$. This implies that the solution

$$\varphi(t, x_0 + x, \omega_0 + \omega) = e^{tR}(x_0 + x) + \int_0^t e^{R(t-s)} B \left[\omega_0(s) + \omega(s)\right] ds, t \geq 0,$$

satisfies for k large enough by (3.3-6) and linearity,

$$\begin{aligned}
&\|\varphi(\tau, x_0 + x, \omega_0 + \omega) - x_0\| \\
&= \left\| e^{\tau R}(x_0 + x) + \int_0^\tau e^{R(\tau - s)} B\left[\omega_0(s) + \omega(s)\right] ds - x_0 \right\| \\
&\leq \left\| e^{\tau R}(x_0 + x) - e^{\tau R}(x_0 + \lambda) \right\| + \left\| e^{\tau R}(x_0 + \lambda) + \int_0^\tau e^{R(\tau - s)} B\left[\omega_0(s) + \omega(s)\right] ds - x_0 \right\| \\
&\leq \left\| e^{\tau R} x - e^{\tau R} \lambda \right\| + \|\varphi(\tau, x_0 + \lambda, \omega_0 + \omega) - x_0\| \\
&\leq c e^{(\rho_j + \delta) k \tau_0} \frac{b_0}{M_j(k\tau_0)} \sqrt{d_j} \leq b_0.
\end{aligned}$$

Hence we have proved that $\prod_{j=1}^r M_j(\tau)^{d_j}$ control functions are sufficient to steer the solutions from all states in $x_0 + \mathcal{C}$ back to $x_0 + \mathcal{C}$ in time τ. By the no-return property of control sets it follows that the trajectories do not leave D within the time interval $[0, \tau]$. By iterated concatenation of these control functions we can construct an $(n\tau, K, D)$-spanning set $\mathcal{S}$ for each $n \in \mathbb{N}$ with cardinality

$$\left(\prod_{j=1}^r M_j(\tau)^{d_j} \right)^n = \left(\prod_{j:\rho_j \geq 0} (\lfloor e^{(\rho_j + \xi)\tau} \rfloor + 1)^{d_j} \right)^n.$$

It follows that

$$\begin{aligned}
\log a_{n\tau}(f, K, Q) &\leq \log \sum_{\omega \in \mathcal{S}} e^{(S_{n\tau} f)(\omega)} = \log \sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega(t)) dt} \\
&= \log \sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega_0(t)) dt + \int_0^{n\tau} [f(\omega(t)) - f(\omega_0(t))] dt} \\
&\leq \log \left[\sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega_0(t)) dt} \cdot e^{\int_0^{n\tau} \varepsilon dt} \right].
\end{aligned}$$

This implies

$$
\begin{aligned}
\frac{1}{n\tau}\log a_{n\tau}(f,K,Q) &\leq \frac{1}{\tau}\sum_{j:\rho_j\geq 0} d_j \log(\lfloor e^{(\rho_j+\xi)\tau}\rfloor + 1) + \frac{1}{n\tau}\int_0^{n\tau} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{1}{\tau}\sum_{j:\rho_j\geq 0} d_j \log(2e^{(\rho_j+\xi)\tau}) + \frac{1}{\tau_0}\int_0^{\tau_0} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{d}{\tau}\log 2 + \frac{1}{\tau}\sum_{j:\rho_j\geq 0} d_j(\rho_j+\xi)\tau + \frac{1}{\tau_0}\int_0^{\tau_0} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{d\xi}{\tau} + \sum_{j:\rho_j\geq 0} d_j\rho_j + \frac{1}{\tau_0}\int_0^{\tau_0} f(\omega_0(t))dt + 2\varepsilon \\
&< S_0 + \frac{1}{\tau_0}\int_0^{\tau_0} f(\omega_0(t))dt + 2\varepsilon.
\end{aligned}
$$

Here we have also used (3.3-5). Since ε can be chosen arbitrarily small and S_0 arbitrarily close to $\sum_{j=1}^{r}\max(0, d_j\rho_j)$, the assertion of the theorem follows. □

Remark 3.3.4. *The Theorem 3.3.3 can be generalized for a control set D with compact closure of a continuous-time control system $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ on a connected smooth manifold M in the following way: Let $D \subset M$ be a control set with nonempty interior and compact closure for the control system Σ. Then for every compact set $K \subset D$, the pair (K, D) is admissible and for all potentials $f \in C(U, \mathbb{R})$ the invariance pressure satisfies*

$$
P_{\mathrm{inv}}(f,K,D) \leq \inf_{(T,x,\omega)} \left\{ \sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x,\omega)\rho_j(x,\omega)\} + \frac{1}{T}\int_0^T f(\omega(s))ds \right\},
$$

where the infimum is taken over all $(T, x, \omega) \in (0, \infty) \times \mathrm{int}D \times \mathcal{U}$ such that the T-periodic controlled trajectory $(\varphi(\cdot, x, \omega), \omega(\cdot))$ is regular on $[0, T]$ (see [27, Sect. 1.5]), the values $\omega(t), t \in [0, T]$, are in a compact subset of $\mathrm{int}U$, and $\rho_1(x, \omega), \ldots, \rho_r(x, \omega)$, $r = r(x, \omega)$, are the different Lyapunov exponents of $(\varphi(\cdot, x, \omega), \omega(\cdot))$ at (x, ω) with corresponding multiplicities $d_1(x, \omega), \ldots, d_r(x, \omega)$ (see [27, Sect. 5.1]). The proof follows the same ideas of [25, Theorem 4.3].

The following remark is helpful to see the relation to Floquet exponents.

Remark 3.3.5. *Consider a τ_0-periodic solution of*

$$
\dot{x}(t) = Ax(t) + Bu(t).
$$

Then the Floquet exponents of the linearized system (linearized with respect to x) are given by the real parts of the eigenvalues of A and also the algebraic multiplicities coincide. More generally, the Lyapunov exponents are given by

$$\lim_{t\to\infty}\frac{1}{t}\log\|D_x\varphi(t,x,u)y\| = \lim_{n\to\infty}\frac{1}{nT}\log\left\|e^{AnT}y\right\| = \lambda,$$

depending on y. In fact, we have to analyze the eigenvalues of the linearization of the map $x \mapsto \varphi(\tau_0,x,u) = e^{A\tau_0}x + \int_0^{\tau_0} e^{A(\tau_0-s)}Bu(s)ds$ given by $D_x\varphi(\tau_0,x,u) = e^{A\tau_0}$. Thus the assertion is a consequence of the spectral mapping theorem.

The Remark 3.3.5 shows that

$$\sum_{\lambda\in\sigma(A)}\max\{0,n_\lambda\mathrm{Re}(\lambda)\} = \sum_{j=1}^{r}\max\{0,d_j\rho_j\},$$

where $\rho_1,\ldots,\rho_r$ are the different Lyapunov exponents with corresponding multiplicities of a periodic solution corresponding to a periodic control. This is the term occurring in the estimate for the invariance entropy in Kawan [27, Theorem 5.1].

Corollary 3.3.6. *Consider a linear control system of the form Σ_{lin} and assume that the pair (A,B) is controllable, that A is hyperbolic and the control range U is a compact neighborhood of the origin. Let D be the unique control set, let $f \in C(U,\mathbb{R})$ and suppose that $\min_{\omega\in U} f(\omega) = f(\omega_0)$ with $\omega_0 \in \mathrm{int}U$ and there exists $x_0 \in \mathrm{int}D$ with $Ax_0 + B\omega_0 = 0$.*

Then for every compact set $K \subset D$ with nonempty interior we have that (K,D) is an admissible pair and

$$P_{\mathrm{inv}}(f,K,D) = \sum_{\lambda\in\sigma(A)}\max\{0,n_\lambda Re(\lambda)\} + f(\omega_0).$$

Proof. This follows by Theorem 3.3.3, since (ω_0,x_0) is a (trivial) periodic solution in $\mathrm{int}U \times \mathrm{int}D$, and for every T-periodic control $\omega(\cdot)$

$$\frac{1}{T}\int_0^T f(\omega(s))ds \ge f(\omega_0).$$

□

Example 3.3.7. *Consider the one-dimensional linear control system given by the differential*

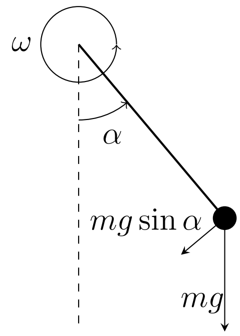


Figure 3.1: Pendulum.

equations

$$\Sigma : \ \dot{x}(t) = ax(t) + \omega(t), \ \omega \in \mathcal{U},$$

where $a > 0$. *We assume that the control range* $U = [-1, 1]$. *Then the compact interval* $Q = \left[-\frac{1}{a}, \frac{1}{a}\right]$ *is the closure of the unique control set with nonempty interior* $D = \mathcal{O}^-(0) = \left(-\frac{1}{a}, \frac{1}{a}\right)$ *of* Σ.

Let $f \in C(U, \mathbb{R})$ *such that* $f(u_0) = \inf f$ *for some* $u_0 \in \mathrm{int}U$. *Then* $x_0 := -\frac{u_0}{a} \in \mathrm{int}D$ *and* (x_0, u_0) *satisfies* $ax_0 + u_0 = 0$. *By Corollary 3.3.6 we have*

$$P_{\mathrm{inv}}(f, K, Q) = \inf f + a.$$

The next example (cf. [36, Section 1.2]) presents an application of outer invariance pressure to a mechanical control system and shows that, in this case, this quantity is related with the exponential growth rate of total impulse of external forces acting on the system.

Example 3.3.8. *Consider a pendulum to which one can apply a torque as an external force (see Figure 3.1). We assume that friction is negligible, that all of the mass is concentrated at the end, and that the rod has unit length. From Newton's law for rotating objects, there results, in terms of the variable* α *that describes the counter clockwise angle with respect to the vertical, the second-order nonlinear differential equation*

$$m\ddot{\alpha}(t) + mg\sin(\alpha(t)) = \omega(t), \tag{3.3-7}$$

where m *is the mass,* g *the acceleration due to gravity, and* $u(t)$ *the value of the external torque at time* t *(counter clockwise being positive).*

The vertical stationary position $(\alpha, \dot{\alpha}) = (\pi, 0)$ *is an equilibrium when the null control*

$\omega_0 \equiv 0$ is applied, but a small deviation from this will result in an unstable motion. Let us assume that our objective is to apply torques as needed to correct such deviations. For small $\alpha - \pi$,

$$\sin(\alpha) = -(\alpha - \pi) + r(\alpha - \pi),$$

when $r(t)$ is a function which satisfies $\lim_{t\to 0} \frac{r(t)}{t} = 0$.

Since only small deviations are of interest, we drop the nonlinear part represented by the term $r(t)$. Thus, with $\gamma := \alpha - \pi$ as a new variable, we replace equation (3.3-7) by the linear differential equation

$$m\ddot{\gamma}(t) - mg\gamma(t) = \omega(t).$$

If we denote $x_1 = \gamma$ and $x_2 = \dot{\gamma}$, then we obtain

$$\Sigma_1 \ : \ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{=:B} \omega, \ \omega(t) \in U := [-\varepsilon, \varepsilon], \varepsilon > 0.$$

Note that the eigenvalues of A are $\lambda_{\pm} = \pm\sqrt{g}$. System Σ_1 is via the (time-invariant) conjugacy map (T, id_U) conjugate to (cf. Example 2.2.11)

$$\Sigma_2 \ : \ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\sqrt{g} & 0 \\ 0 & \sqrt{g} \end{bmatrix}}_{=:\widetilde{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2m} \\ \frac{1}{2m} \end{bmatrix}}_{=:\widetilde{B}} \omega,$$

because $\widetilde{A} = TAT^{-1}$ and $\widetilde{B} = TB$, where

$$T = \frac{1}{2} \begin{bmatrix} -\sqrt{g} & 1 \\ \sqrt{g} & 1 \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} -\frac{1}{\sqrt{g}} & \frac{1}{\sqrt{g}} \\ 1 & 1 \end{bmatrix}.$$

Note that $\widetilde{A}$ is hyperbolic and the pair $(\widetilde{A}, \widetilde{B})$ is controllable. By Theorem 1.3.10, the unique control set $\tilde{D}$ with nonvoid interior of Σ_2 is

$$\tilde{D} = \overline{\mathcal{O}^+(0)} \cap \mathcal{O}^-(0) = \left[-\frac{\varepsilon}{2m\sqrt{g}}, \frac{\varepsilon}{2m\sqrt{g}}\right] \times \left(-\frac{\varepsilon}{2m\sqrt{g}}, \frac{\varepsilon}{2m\sqrt{g}}\right).$$

Then the unique control set with nonvoid interior of Σ_1 is given by $D := T(\tilde{D})$ and one *computes*

$$D = [-d, d] \times (-d, d) \text{ with } d := \varepsilon \frac{\sqrt{g}+1}{2m\sqrt{g}}.$$

Here for a compact subset $K \subset D$ a (τ, K, D)-spanning set $\mathcal{S}$ represents a set of external torques ω that cause the angular position of the pendulum to remain in the interval $[-d, d]$ and such that its angular velocity does not exceed $(-d, d)$ when it starts in K. If $f(u) = |u|$, $u \in U = [-\varepsilon, \varepsilon]$, then $f \in C(U, \mathbb{R})$ and $0 = f(0) = \inf f$. Note that here $(S_\tau f)(\omega)$ represents the impulse of the torque ω until time τ. Hence, the invariance pressure $P_{\text{inv}}(f, K, D)$ measures the exponential growth rate of the quantity of total impulse required of the external torques acting on the system to remain in D as time tends to infinity. Corollary 3.3.6 implies that $P_{\text{inv}}(f, K, D) = \sqrt{g} = h_{\text{inv}}(K, D)$. The reason is that within the control set D one may steer the system from K arbitrarily close to the equilibrium $0 \in \mathbb{R}^2$ and keep it there with arbitrarily small torque.

3.4 Lower Bounds

Consider a control system $\Sigma = (\mathbb{R}, M, U, \mathcal{U}, \varphi)$ on a Riemannian manifold (M, g) and suppose that for each $t \geq 0$ and $\omega \in \mathcal{U}$, the map $\varphi_{t,\omega} : M \to M$ is a diffeomorphism. In our case, it happens if the following two assumptions are verified:

- For each $u \in U$, the map $F_u : M \to TM$ is infinitely differentiable.
- $\mathcal{U} = \{\omega \in L^\infty(\mathbb{R}, \mathbb{R}^m);\ \omega(t) \in U \text{ a.e.}\}$.

The ideas of this section come from [26] and [27, Section 4.2]. Here we provide a lower bound for the invariance pressure when both sets of an admissible pair (K, Q) have finite and positive Riemannian volume vol.

Theorem 3.4.1. *Let $f \in C(U, \mathbb{R})$ and (K, Q) an admissible pair for Σ such that Q is open or closed and both K and Q have finite and positive volume. Then*

$$P_{\text{inv}}(f, K, Q) \quad \geq \quad \limsup_{\tau \to \infty} \frac{1}{\tau} \left(\inf_{\substack{(x,\omega) \in K \times \mathcal{U} \\ \varphi([0,\tau], x, \omega) \subset Q}} \int_0^\tau f(\omega(s)) ds + \log \max \left\{ 1, \inf_{\substack{(x,\omega) \in K \times \mathcal{U} \\ \varphi([0,\tau], x, \omega) \subset Q}} |\det \mathrm{d}_x \varphi_{\tau,\omega}| \right\} \right).$$

Proof. Initially note that we may assume that for all $\tau > 0$, there exists a finite (τ, K, Q)-spanning set, because if there is $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ every (τ, K, Q)-spanning set is infinity, then $P_{\text{inv}}(f, K, Q) = \infty$ and the result becomes trivial.

Let $\tau > 0$, $\mathcal{S}$ be a finite (τ, K, Q)-spanning set. For each $\omega \in \mathcal{S}$, define

$$K_\omega := \{x \in K;\ \varphi([0,\tau], x, \omega) \subset Q\}.$$

In this case, $K = \cup_{\omega\in\mathcal{S}} K_\omega$, because $\mathcal{S}$ is a (τ, K, Q)-spanning. If Q is closed, we can write each K_ω as countable intersection of measurable sets:

$$K_\omega = K \cap \bigcap_{t\in[0,\tau]\cap\mathbb{Q}} \varphi_{t,\omega}^{-1}(Q),$$

and if Q is open, K_ω is relatively open in Q by continuity of $\varphi_\omega(\cdot,\cdot)$. Since $\varphi_{\tau,\omega}$ is a diffeomorphism, also $\varphi_{\tau,\omega}(K_\omega)$ is measurable. Then

$$\mathrm{vol}(\varphi_{\tau,\omega}(K_\omega)) = \int_{\varphi_{\tau,\omega}(K_\omega)} \mathrm{dvol} = \int_{K_\omega} |\det \mathrm{d}\varphi_{\tau,\omega}| \mathrm{dvol} \geq \mathrm{vol}(K_\omega) \underbrace{\inf_{\substack{(x,\omega)\in K\times\mathcal{U} \\ \varphi([0,\tau],x,\omega)\subset Q}} |\det \mathrm{d}_x\varphi_{\tau,\omega}|}_{\alpha(\tau)}.$$

Put $\beta(\tau) := \inf_{\substack{(x,\omega)\in K\times\mathcal{U} \\ \varphi([0,\tau],x,\omega)\subset Q}} (S_\tau f)(\omega)$, then

$$\begin{aligned} e^{\beta(\tau)}\mathrm{vol}(K) &\leq \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}\mathrm{vol}(K_\omega) \leq \max_{\omega\in\mathcal{S}} \mathrm{vol}(K_\omega) \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} \\ &\leq \frac{\mathrm{vol}(Q)}{\max\{1, \alpha(\tau)\}} \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}, \end{aligned}$$

because $\varphi_{\tau,\omega}(K_\omega) \subset Q$, implying $\mathrm{vol}(\varphi_{\tau,\omega}(K_\omega)) \leq \mathrm{vol}(Q)$. The previous inequalities hold for every (τ, K, Q)-spanning set $\mathcal{S}$, therefore

$$a_\tau(f, K, Q) \geq \frac{\mathrm{vol}(K)}{\mathrm{vol}(Q)} e^{\beta(\tau)} \max\{1, \alpha(\tau)\},$$

The result follows from the previous inequality and by the fact that $\frac{\mathrm{vol}(K)}{\mathrm{vol}(Q)} \in (0,\infty)$. $\square$

Corollary 3.4.2. *Under the same assumptions of Theorem 3.4.1, the following estimates hold:*

i)

$$\begin{aligned} &P_{\mathrm{inv}}(f, K, Q) \\ \geq\ & \limsup_{\tau\to\infty} \frac{1}{\tau} \left(\inf_{\substack{(x,\omega)\in K\times\mathcal{U} \\ \varphi([0,\tau]x,\omega)\subset Q}} \int_0^\tau f(\omega(s))ds + \inf_{\substack{(x,\omega)\in K\times\mathcal{U} \\ \varphi([0,\tau]x,\omega)\subset Q}} \max\left\{0, \int_0^\tau \mathrm{div} F_{\omega(s)}(\varphi(s,x,\omega))ds\right\}\right). \end{aligned}$$

ii)

$$
\begin{aligned}
P_{\text{inv}}(f,K,Q) \;&\geq\; \liminf_{\tau\to\infty}\left(\inf_{\substack{(x,\omega)\in K\times\mathcal{U}\\ \varphi([0,\tau]x,\omega)\subset Q}} \frac{1}{\tau}\int_0^\tau f(\omega(s))ds\right)\\
&+\; \liminf_{\tau\to\infty}\left(\inf_{\substack{(x,\omega)\in K\times\mathcal{U}\\ \varphi([0,\tau]x,\omega)\subset Q}} \frac{1}{\tau}\max\left\{0,\int_0^\tau \mathbf{div}F_{\omega(s)}(\varphi(s,x,\omega))ds\right\}\right).
\end{aligned}
$$

iii)

$$P_{\text{inv}}(f,K,Q) \geq \max\left\{0, \inf_{(x,u)\in Q\times U}\mathbf{div}F_u(x)\right\} + \inf f.$$

Proof. i) The result follows immediately from the Liouville's formula

$$\log|\det \mathbf{d}_x\varphi_{\tau,\omega}| = \int_0^\tau \mathbf{div}F_{\omega(s)}(\varphi(s,x,\omega))ds.$$

ii) It is sufficient to note that, in general, given functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$, then

$$\limsup_{\tau\to\infty} g_1(\tau) \geq \liminf_{\tau\to\infty} g_1(\tau) \text{ and } \liminf_{\tau\to\infty}(g_1(\tau)+g_2(\tau)) \geq \liminf_{\tau\to\infty} g_1(\tau)+\liminf_{\tau\to\infty} g_2(\tau).$$

iii) We just have to observe that

$$\tau \inf_{(x,u)\in Q\times U}\mathbf{div}F_u(x) \leq \int_0^\tau \mathbf{div}F_{\omega(s)}(\varphi(s,x,\omega))ds \text{ and } \tau\inf f \leq \int_0^\tau f(\omega(s))ds.$$

□

CHAPTER 4

INVARIANCE PRESSURE FOR DISCRETE-TIME CONTROL SYSTEMS

In 2004, Nair et al. defined the topological feedback entropy via invariant covers of a set Q (with some invariant conditions imposed) and they proved that this quantity characterizes the smallest average data rate above which it is possible to render the set Q invariant by a causal coding and control law (see [32, Section III]). Nine years later, Colonius, Kawan and Nair modeled rightly the original definition of invariance entropy presented in [9] via spanning sets for discrete-time control systems and they showed in [10] that these control entropies are essentially equivalent when Q is strongly invariant.

This equivalence of the definitions via spanning sets and (invariant) covers is also verified for topological entropy for dynamical systems on compact metric spaces (see, for instance, [39, Theorem 7.8]). Although this fact is valid for topological pressure only when one consider covers with diameter arbitrarily small.

This chapter follows [6] and proposes to generalize the definitions and some results presented in [32] and [10] for the invariance pressure case and, in contrast to what occurs with topological pressure, we can show that for all $f \in C(U, \mathbb{R})$, both definitions via spanning sets (inner invariance pressure) and invariant covers (topological feedback pressure) coincide for a strongly invariant set Q. In the last section we propose a generalization of the concept of transmission data rate of a channel, presented in [32], and we get that the invariance pressure is characterized by the smallest weighted average data rate above which it is possible to render the set Q invariant by a causal coding and control law.

4.1 Inner Invariance Pressure

In order to construct the inner invariance pressure for discrete-time control systems let, for $f \in C(U, \mathbb{R})$,

$$(S_n f)(\omega) := \sum_{i=0}^{n-1} f(u_i), \quad \omega = (u_i)_{i \in \mathbb{N}_0} \in \mathcal{U},$$

and

$$a_n(f, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \text{ is a } (n, Q, \mathit{int}Q)\text{-spanning} \right\}.$$

Definition 4.1.1. *For a discrete-time control system of the form (1.3-5), a strongly invariant compact set $Q \subset X$ and $f \in C(U, \mathbb{R})$ the* **inner invariance pressure** *of Q is defined by the limit*

$$P_{\text{inv,int}}(f, Q) = \lim_{n \to \infty} \frac{1}{n} \log a_n(f, Q). \tag{4.1-1}$$

Note that if $f = \mathbf{0}$ is the null function in $C(U, \mathbb{R})$, then $\sum_{\omega \in \mathcal{S}} e^{(S_n \mathbf{0})(\omega)} = \sum_{\omega \in \mathcal{S}} 1 = \#\mathcal{S}$, hence

$$\begin{aligned} a_n(\mathbf{0}, Q) &= \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n \mathbf{0})(\omega)};\ \mathcal{S} \text{ is a } (n, Q, \mathit{int}Q)\text{-spanning} \right\} \\ &= \inf \{ \#\mathcal{S};\ \mathcal{S} \text{ is a } (n, Q, \mathit{int}Q)\text{-spanning} \} \\ &= r_{\text{inv,int}}(n, Q). \end{aligned} \tag{4.1-2}$$

Taking the logarithm, dividing by n and letting n tend to ∞ one finds that $P_{\text{inv,int}}(\mathbf{0}, Q) = h_{\text{inv,int}}(Q)$. Hence the inner invariance pressure generalizes the inner invariance entropy.

Remark 4.1.2. *The same result of Proposition 2.1.4 is verified for $a_n(f, Q)$, that is,*

$$a_n(f, Q) = \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \textit{ is a finite } (n, Q, \text{int}Q)\textit{-spanning} \right\}.$$

The proof is analogous to that made in Proposition 2.1.4 and can be found in [6, Proposition 5].

Based on this result, in the following we will only consider finite spanning sets. We still have to show that the limit in (4.1-1) actually exists.

Proposition 4.1.3. *For $f \in C(U, \mathbb{R})$, the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log a_n(f,Q) = \inf_{n\geq 1} \frac{1}{n} \log a_n(f,Q).$$

Proof. This follows by a standard lemma in this context (cf., Lemma 1.4.9), if we can show that the sequence $\log a_n(f,Q), n \in \mathbb{N}$, is subadditive. Let $\mathcal{S}_1$ be a $(n, Q, \text{int}Q)$-spanning set and $\mathcal{S}_2$ a $(k, Q, \text{int}Q)$-spanning set. Then define control sequences of length $n+k$ by

$$\omega := (u_0, \ldots, u_{n-1}, v_0, \ldots, v_{k-1}) \in U^{n+k}.$$

for each $\omega_1 = (u_0, \ldots, u_{n-1}) \in \mathcal{S}_1$ and $\omega_2 = (v_0, \ldots, v_{k-1}) \in \mathcal{S}_2$. We claim that the set $\mathcal{S}$ of these control sequences is a $(n+k, Q, \text{int}Q)$-spanning. In fact, for $x \in Q$ there exist $\omega_1 \in \mathcal{S}_1$ such that

$$\varphi(j, x, \omega) = \varphi(j, x, \omega_1) \in \text{int}Q, j = 1, \ldots, n.$$

Since $\varphi(n, x, \omega_1) \in \text{int}Q \subset Q$ and $\mathcal{S}_2$ is strongly $(k, Q, \text{int}Q)$-spanning, there is a $\omega_2 \in \mathcal{S}_2$ such that

$$\varphi(n+j, x, \omega) = \varphi(j, \varphi(n, x, \omega_1), \omega_2) \in \text{int}Q, j = 1, \ldots, k.$$

This shows the claim. Furthermore, for all $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\sum_{\omega\in\mathcal{S}} e^{(S_{n+k}f)(\omega)} = \sum_{\omega\in\mathcal{S}} e^{(S_n f)(\omega_1)} e^{(S_k f)(\omega_2)} \leq \sum_{\omega_1\in\mathcal{S}_1} e^{(S_n f)(\omega_1)} \sum_{\omega_2\in\mathcal{S}_2} e^{(S_k f)(\omega_2)}.$$

Hence $a_{n+k}(f,Q) \leq a_n(f,Q) a_k(f,Q)$ and the subadditivity property follows proving the assertion. □

Example 4.1.4. *Consider a scalar linear control system of the form*

$$x_{k+1} = a x_{k+1} + u_k, u_k \in U := [-1, 1],$$

with $a > 1$ and let $Q := \left[-\frac{1}{a-1} + \varepsilon, \frac{1}{a-1} - \varepsilon\right]$, where $\varepsilon > 0$ is small. Let $f \in C(U, \mathbb{R})$ be given by $f(u) = |u|, u \in [-1, 1]$. We claim that $P_{\text{inv,int}}(f, Q) = \log a = h_{\text{inv,int}}(Q)$, where the equality for the inner invariance entropy of Q has been shown in Colonius, Kawan and Nair [10, Example 3.2].

In order to show $P_{\text{inv,int}}(f, Q) \geq \log a$, consider for $n \in \mathbb{N}$ a finite $(n, Q, intQ)$-spanning set $\mathcal{S}$. For $\omega \in \mathcal{S}$ define

$$Q_\omega := \{x \in Q; \varphi(j, x, \omega) \in \text{int}Q \textit{ for } j = 1, \ldots, n\}.$$

Then $Q = \bigcup_{\omega \in \mathcal{S}} Q_\omega$ and hence the Lebesgue measure λ satisfies $\lambda(Q) \leq \sum_{\omega \in \mathcal{S}} \lambda(Q_\omega)$. Furthermore, for $x \in Q_\omega$ we have

$$\varphi(n, x, \omega) = a^n x + \sum_{i=0}^{n} a^i u_i \in Q,$$

which implies that $\lambda(Q) \geq a^n \lambda(Q_\omega)$. Thus

$$\lambda(Q) \leq \sum_{\omega \in \mathcal{S}} \lambda(Q_\omega) \leq \#\mathcal{S} \cdot \max_{\omega \in \mathcal{S}} \lambda(Q_\omega) \leq \#\mathcal{S} \cdot a^{-n} \lambda(Q)$$

and hence $\#\mathcal{S} \geq a^{n-1}$. Since $f(u) \geq 0$, it follows that

$$a_n(f, Q) = \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \textit{ is a } (n, Q, \text{int}Q)\textit{-spanning set} \right\} \geq a^n$$

and hence

$$P_{\text{inv,int}}(f, Q) = \lim_{n \to \infty} \frac{1}{n} \log a_n(f, Q) \geq \log a.$$

In order to prove $P_{\text{inv,int}}(f, Q) \leq \log a$, we use that the inner invariance entropy is given by $h_{\text{inv,int}}(Q) = \log a$. If a solution with $x_0 \in Q$ and control values $u_i \in U$ satisfies for $k \geq 1$

$$\varphi(k, x_0, \omega) = a^k x_0 + \sum_{i=0}^{k-1} a^i u_i \in \text{int}Q,$$

then it follows for every $\delta \in (0, 1)$ that $\delta u_i \in \delta U = [-\delta, \delta] \subset [-1, 1] = U$ for all i and

$$\delta\varphi(k, x_0, \omega) = a^k \delta x_0 + \sum_{i=0}^{k-1} a^i \delta u_i \in \text{int}(\delta Q) \subset \text{int}(Q).$$

Hence the solution keeps the initial point $\delta x_0 \in \delta Q$ with control values $\delta u_i \in \delta U$ in $\text{int}(\delta Q)$. Observe that $f(\delta u_i) = |\delta u_i| \leq \delta$.

Take $0 < \delta < \frac{1}{a-1} - \varepsilon$. Then for $x_0 \in Q = \left[-\frac{1}{a-1} + \varepsilon, \frac{1}{a-1} - \varepsilon\right]$ there are $n \in \mathbb{N}$ and

$\omega = (u_i)$ with $u_i \in U = [-1, 1]$ such that

$$|\varphi(n, x_0, \omega)| \leq \delta \text{ and } \varphi(k, x_0, \omega) \in Q \text{ for all } k = 1, \ldots, n-1.$$

This is seen as follows. If $x_0 \in \left[0, \frac{1}{a-1} - \varepsilon\right]$, we can make a step to the left of x_0 of length l where $l \in (0, (a-1)\varepsilon]$ is arbitrary. In fact, using the control value $u_0 = -1 \in [-1, 1]$ one obtains for $x_1 = ax_0 + u_0$ that

$$x_1 - x_0 = ax_0 - x_0 - 1 \leq (a-1)\left(\frac{1}{a-1} - \varepsilon\right) - 1 = -(a-1)\varepsilon < 0.$$

Similarly, for $u_0 = -1 + (a-1)\varepsilon \in [-1, 1]$, one computes $x_1 = x_0$ and hence, by continuity, one can make steps of length l to the left.

Analogously for $x_0 \in \left[\frac{1}{1-a} + \varepsilon, 0\right]$ one can make steps to the right.

Going several steps, if necessary, one can reach the interval $(-\delta, \delta)$ from each point of Q.

By the arguments above we know that we can stay in the interval $(-\delta, \delta)$. Together we have shown that there is a time $n_0 \in \mathbb{N}$ such that for every $x \in Q$ there is a control ω with $\varphi(n_0, x, \omega) \in (-\delta, \delta)$. By continuity, there are finitely many controls $\omega_1, \ldots, \omega_N$ such that for every $x \in Q$ there is ω_i with $\varphi(n_0, x, \omega_i) \in (-\delta, \delta)$.

Now choose a finite $(n, Q, intQ)$-spanning set $\mathcal{S}$ with minimal cardinality $\#\mathcal{S} = r_{\text{inv,int}}(n, Q)$. This yields the set $\mathcal{S}_\delta := \{\delta\omega; \omega \in \mathcal{S}\}$ of controls with values in $[-\delta, \delta]$ which keep every element in δQ. Concatenations of the controls in $\mathcal{S}_\delta$ with the controls $\omega_1, \ldots, \omega_N$ yields an $(n_0+n, Q, intQ)$-spanning set $\mathcal{S}'$ with cardinality $\#\mathcal{S}' \leq N \cdot \#\mathcal{S}$. For $k \in \{n_0+1, \ldots, n_0+n\}$, the controls in $\mathcal{S}'$ have values in $[-\delta, \delta]$, hence $f(u) = |u| \leq \delta$ here.

We compute for $\omega' = (u_i) \in \mathcal{S}'$

$$\begin{aligned}(S_{n_0+n}f)(\omega') &= \sum_{i=0}^{n_0+n-1} f(u_i) = \sum_{i=0}^{n_0-1} f(u_i) + \sum_{i=n_0}^{n_0+n-1} f(u_i) \\ &\leq n_0 \max_{u \in [-1,1]} |u| + n \max_{u \in [-\delta,\delta]} |u| = n_0 + n\delta.\end{aligned}$$

This yields

$$\begin{aligned}a_{n+n_0}(f, Q) &\leq \sum_{\omega' \in \mathcal{S}'} e^{(S_{n+n_0}f)(\omega)} \leq \#\mathcal{S}' \cdot e^{n_0+n\delta} \leq N \cdot \#\mathcal{S} \cdot e^{n_0+n\delta} \\ &= N \cdot r_{\text{inv,int}}(n, Q) \cdot e^{n_0+n\delta},\end{aligned}$$

and hence

$$\begin{aligned}
P_{\text{inv,int}}(f,Q) &= \limsup_{n\to\infty} \frac{1}{n+n_0} \log a_{n+n_0}(f,Q) \\
&\leq \limsup_{n\to\infty} \left[\frac{1}{n+n_0} \log N + \frac{n}{n+n_0} \frac{1}{n} \log r_{inv,int}(n,Q) + \frac{n_0+n\delta}{n+n_0} \right] \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log r_{\text{inv,int}}(n,Q) + \limsup_{n\to\infty} \frac{n_0+n\delta}{n+n_0}.
\end{aligned}$$

Since $\frac{n_0+n\delta}{n+n_0} \leq 2\delta$ *for* n *large enough it follows that* $P_{\text{inv,int}}(f,Q) \leq h_{\text{inv,int}}(Q) + 2\delta$ *which implies* $P_{\text{inv,int}}(f,Q) \leq h_{\text{inv,int}}(Q)$*, since* $\delta > 0$ *is arbitrary.*

The discrete-time case has a particular formulation of Proposition 2.2.7. Let $\Sigma = (\mathbb{Z}, X, U, \mathcal{U}, \varphi)$ and suppose we take $N \in \mathbb{N}$ steps at once. Then, naturally, the solution $\varphi(N,x,\omega)$ may be in Q while there may exist $i \in \{1, ..., N-1\}$ with $\varphi(i,x,\omega) \notin Q$. Hence, for a power rule in invariance problems of discrete-time control systems one has to exclude this a-priori. Therefore, we must assume that Q satisfies the no-return property for the discrete-time environment.

Starting from control system (1.3-5) define the following control system. Given $N \in \mathbb{N}$, the control range is $U^N = U \times \ldots \times U$ and the set of corresponding controls is denoted by $\mathcal{U}^N$. Then a bijective relation between the controls in $\mathcal{U}$ and in $\mathcal{U}^N$ is given by

$$i : \mathcal{U} \to \mathcal{U}^N : \omega = (\omega_k) \mapsto (\omega_k^N) := (\omega(Nk), \ldots, \omega(Nk+N-1)).$$

The solutions will be given by $\varphi^N(0,x,\omega) = x$ and for $k \geq 1$

$$\varphi^N(k,x,i(\omega)) = \varphi(kN,x,\omega).$$

Then, these are the solutions of $\Sigma_N = (\mathbb{Z}, X, U^N, \mathcal{U}^N, \varphi^N)$ whose difference equation has the form

$$x_{k+1} = F^{(N)}(x_k, v_k),\ v_k \in U^N, \tag{4.1-3}$$

and the solutions can be written as

$$\varphi^N(k,x,\omega) = \varphi_{N,\theta^{N(k-1)}(\omega)} \circ \cdots \circ \varphi_{N,\omega}(x).$$

Proposition 4.1.5. *In the above setting, let* Q *a strongly invariant set for* Σ *which* Q *satisfies* *the no-return property. Then Q is a strongly invariant set for Σ_N and for every $f \in C(U,\mathbb{R})$*

$$P_{\text{inv,int}}(g,Q;\Sigma_N) = N \cdot P_{\text{inv,int}}(f,Q;\Sigma),$$

where $g \in C(U^N,\mathbb{R})$ is given by $g(u_0,\ldots,u_{N-1}) := \sum_{i=0}^{N-1} f(u_i)$.

Proof. It is easy to see that Q is a strongly invariant set for Σ_N. Note also that if $\mathcal{S} \subset \mathcal{U}$ is a $(nN,Q,\text{int}Q)$-spanning set for Σ, then $\mathcal{S}^N := \{i(\omega);\ \omega \in \mathcal{S}\}$ is a $(n,Q,\text{int}Q)$-spanning set for Σ_N. Analogously, if $\mathcal{S}^N$ is a $(n,Q,\text{int}Q)$-spanning set for Σ_N, then $i^{-1}(\mathcal{S}^N)$ is a $(nN,Q,\text{int}Q)$-spanning set for Σ. Therefore

$$\sum_{\omega\in\mathcal{S}^N} e^{(S_n g)(\omega)} = \sum_{\omega\in i^{-1}(\mathcal{S}^N)} e^{(S_{nN} f)(\omega)}.$$

Then $a_n(g,Q;\Sigma_N) = a_{nN}(f,Q;\Sigma)$ and so

$$\begin{aligned} P_{\text{inv,int}}(g,Q;\Sigma_N) &= \lim_{n\to\infty}\frac{1}{n}\log a_n(g,Q;\Sigma_N) = N\lim_{n\to\infty}\frac{1}{nN}\log a_{nN}(f,Q;\Sigma) \\ &= N\cdot P_{\text{inv,int}}(f,Q;\Sigma). \end{aligned}$$

□

4.2 Topological Feedback Pressure

Next we introduce a notion of invariance pressure based on feedbacks and show that it coincides with the invariance pressure defined above.

Open covers in entropy theory of dynamical systems are replaced in case of control systems by invariant open covers, introduced in [32].

We say that a set of controls of the form

$$\mathcal{W}_n = \{\omega(\alpha_i); \alpha_i \in \mathcal{A}^{\mathbb{N}_0} \text{ for } i \in I\}$$

is a **generating set** of feedback controls (of length $n\tau$) for the invariant open cover $\mathcal{C}$, if the sets $B_n(\alpha_i), i \in I$, form a subcover of $\mathcal{B}_n(\mathcal{C})$ which is minimal in the sense that none of its elements may be omitted in order to cover Q. (Its number of elements needs not be minimal among all subcovers.) Hence $Q = \bigcup_{i\in I} B_n(\alpha_i)$ and the number of elements

$\#I$ in the index set I is bounded by $\#\mathcal{B}_n$.

Define for $\omega = (u_i)_{i\in\mathbb{N}_0} \in \mathcal{U}$

$$(S_{n\tau})(\omega) = \sum_{i=0}^{n\tau-1} f(u_i),$$

and set

$$q_n(f, Q, \mathcal{C}) = \inf\left\{\sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau}f)(\omega)}; \mathcal{W}_n \text{ is a generating set for } \mathcal{C}\right\}.$$

Definition 4.2.1. *Consider a discrete-time control system of the form (1.3-5), a strongly invariant compact set $Q \subset X$ and $f \in C(U, \mathbb{R})$. For an invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$, put*

$$P_{\text{fb}}(f, Q, \mathcal{C}) = \lim_{n\to\infty} \frac{1}{n\tau} \log q_n(f, Q, \mathcal{C}) \tag{4.2-4}$$

and the **topological feedback pressure** *is defined as*

$$P_{\text{fb}}(f, Q) = \inf\{P_{\text{fb}}(f, Q, \mathcal{C});\ \mathcal{C} \textit{ is an invariant open cover of } Q\}.$$

Here are several comments on this definition. If $f = \mathbf{0}$ is the null function in $C(U, \mathbb{R})$, then

$$\sum_{\omega\in\mathcal{W}_n} e^{(S_n\mathbf{0})(\omega)} = \sum_{\omega\in\mathcal{W}_n} 1 = \#\mathcal{W}_n,$$

hence

$$\begin{aligned} q_n(\mathbf{0}, Q, \mathcal{C}) &= \inf\left\{\sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau}0)(\omega)}; \mathcal{W}_n \text{ is a generating set for } \mathcal{C}\right\} \\ &= \inf\{\#\mathcal{B};\ \mathcal{B} \text{ is a subcover of } \mathcal{B}_n\} = N(\mathcal{B}_n; Q), \end{aligned}$$

where $N(\mathcal{B}_n; Q)$ denotes the minimal number of elements in a subcover of $\mathcal{B}_n$. Hence one finds that the topological feedback entropy $h_{\text{fb}}(\mathcal{C})$ of $\mathcal{C}$ (as defined in [27, p. 70]) satisfies

$$h_{\text{fb}}(\mathcal{C}) := \lim_{n\to\infty} \frac{1}{n\tau} \log N(\mathcal{B}_n; Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log q_n(\mathbf{0}, Q, \mathcal{C}) = P_{\text{fb}}(\mathbf{0}, \mathcal{C}),$$

and so the topological feedback entropy of the control system (1.3-5) satisfies

$$\begin{aligned} h_{\text{fb}}(Q) &:= \inf\{h_{\text{fb}}(\mathcal{C});\ \ \mathcal{C} \text{ is an invariant open cover of } Q\} \\ &= \inf\{P_{\text{fb}}(\mathbf{0}, \mathcal{C});\ \mathcal{C} \text{ is an invariant open cover of } Q\} = P_{\text{fb}}(\mathbf{0}, Q). \end{aligned}$$

Hence the topological feedback pressure is a generalization of the topological feedback entropy.

The following lemma provides the remaining proof that the limit in (4.2-4) actually exists.

Lemma 4.2.2. *If $f \in C(U, \mathbb{R})$ and $\mathcal{C} = (\mathcal{A}, \tau, G)$ is an invariant open cover of Q, then the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(f, Q, \mathcal{C}) = \inf_{n\geq 1} \frac{1}{n} \log q_n(f, Q, \mathcal{C}).$$

Proof. The assertions will follow from Lemma 1.4.9 if the sequence $\log q_n(f, Q, \mathcal{C})$, $n \in \mathbb{N}$, is subadditive. This will be shown by constructing a generating set $\mathcal{W}_{n+k}$ from generating sets $\mathcal{W}_n$ and $\mathcal{W}_k$ with the desired properties.

Let $\mathcal{W}_n = \{\omega(\alpha_{i_1}), \ldots, \omega(\alpha_{i_M})\}$ and $\mathcal{W}_k = \{\omega(\beta_{i_1}), \ldots, \omega(\beta_{i_K})\}$ be generating sets of feedback controls. Here α_i and β_j are given by sequences of sets in $\mathcal{A}$ in the form $\alpha_i = (A_\sigma^{\alpha_i})_\sigma$ and $\beta_j = \left(A_\sigma^{\beta_i}\right)_\sigma$. Then define for all i and j sequences in $\mathcal{A}$ by

$$\alpha_i\beta_j = \left(A_0^{\alpha_i}, \ldots, A_{n-1}^{\alpha_i}, A_0^{\beta_j}, \ldots, A_{k-1}^{\beta_j}, \ldots\right).$$

If we denote by $A_\sigma^{\alpha_i\beta_j}$ the σth element of $\alpha_i\beta_j$, then

$$A_\sigma^{\alpha_i\beta_j} = \begin{cases} A_\sigma^{\alpha_i}, & \text{if} \quad 0 \leq \sigma \leq n-1 \\ A_{\sigma-n}^{\beta_j}, & \text{if} \quad \sigma \geq n. \end{cases}$$

Claim: The set

$$\{\omega(\alpha_i\beta_j); i \in \{i_1, \ldots, i_M\}, j \in \{j_1, \ldots, j_K\}\} \tag{4.2-5}$$

contains a generating set of feedback controls.

First note that by the cocycle property one finds for $\sigma = 0, \ldots, k$

$$\varphi_{(\sigma+n)\tau,\omega(\alpha_i\beta_j)} = \varphi_{\sigma\tau,(\theta^{n\tau}\omega(\alpha_i\beta_j))} \circ \varphi_{n\tau,\omega(\alpha_i\beta_j)} = \varphi_{\sigma\tau,\omega(\beta_j)} \circ \varphi_{n\tau,\omega(\alpha_i)},$$

and hence

$$\varphi^{-1}_{(\sigma+n)\tau,\omega(\alpha_i\beta_j)} = \varphi^{-1}_{n\tau,\omega(\alpha_i)} \circ \varphi^{-1}_{\sigma\tau,\omega(\beta_j)}.$$

Thus for all i and j

$$B_{n+k}(\alpha_i\beta_j) = B_n(\alpha_i) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i\beta_j)}(B_k(\beta_j)). \tag{4.2-6}$$

In fact,

$$\begin{aligned}
B_{n+k}(\alpha_i\beta_j) &= \bigcap_{\sigma=0}^{n+k-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i\beta_j)}(A^{\alpha_i\beta_j}_\sigma) \\
&= \bigcap_{\sigma=0}^{n-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i\beta_j)}(A^{\alpha_i\beta_j}_\sigma) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i\beta_j)}\left[\bigcap_{\sigma=0}^{k-1} \varphi^{-1}_{\sigma\tau,\theta^{n\tau}\omega(\alpha_i\beta_j)}(A^{\alpha_i\beta_j}_{\sigma+n})\right] \\
&= \bigcap_{\sigma=0}^{n-1} \varphi^{-1}_{\sigma\tau,\omega(\alpha_i)}(A^{\alpha_i}_\sigma) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i\beta_j)}\left[\bigcap_{\sigma=0}^{k-1} \varphi^{-1}_{\sigma\tau,\omega(\beta_j)}(A^{\beta_j}_\sigma)\right] \\
&= B_n(\alpha_i) \cap \varphi^{-1}_{n\tau,\omega(\alpha_i\beta_j)}(B_k(\beta_j)).
\end{aligned}$$

Clearly the sets $B_{n+k}(\alpha_i\beta_j)$ are elements of $\mathcal{B}_{n+k}(\mathcal{C})$. It follows from (4.2-6) that they cover Q, since this is valid for the families

$$\{B_n(\alpha_i); i \in \{i_1, \ldots, i_M\}\} \text{ and } \{B_n(\beta_j); j \in \{j_1, \ldots, j_K\}\}.$$

Hence the collection in (4.2-5) is a subcover of $\mathcal{B}_{n+k}(\mathcal{C})$ and one finds in the family (4.2-5) an associated generating set of feedback controls which we denote by $\mathcal{W}_{n+k}$. Thus the **Claim** is proved.

In order to show subadditivity of the sequence $\log q_n(f, Q, \mathcal{C}), n \in \mathbb{N}$, note that for all $n, k \in \mathbb{N}$

$$\begin{aligned}
\sum_{\omega\in\mathcal{W}_{n+k}} e^{(S_{(n+k)\tau}f)(\omega)} &= \sum_{\omega\in\mathcal{W}_{n+k}} e^{(S_{n\tau}f)(\omega)} e^{(S_{k\tau}f)(\theta^{n\tau}\omega)} \\
&\leq \sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau}f)(\omega)} \sum_{\omega\in\mathcal{W}_k} e^{(S_{k\tau}f)(\omega)}.
\end{aligned}$$

Since $\mathcal{W}_n$ and $\mathcal{W}_k$ are arbitrary it follows that $q_{n+k}(f,Q,\mathcal{C}) \leq q_n(f,Q,\mathcal{C}) \cdot q_k(f,Q,\mathcal{C})$. This implies the required subadditivity concluding the proof. □

Next we show that this feedback invariance pressure coincides with the inner invariance pressure introduced in Definition 4.1.1 and it generalizes the Theorem 1.4.11.

Theorem 4.2.3. *If $f \in C(U,\mathbb{R})$ and Q is a strongly invariant compact subset of X, then*

$$P_{\text{inv,int}}(f,Q) = P_{\text{fb}}(f,Q).$$

Proof. First we prove the inequality $P_{\text{inv,int}}(f,Q) \leq P_{\text{fb}}(f,Q)$. Let $\mathcal{C} = (\mathcal{A},\tau,G)$ be an invariant open cover. Then for $n \in \mathbb{N}$, every generating set $\mathcal{W}_n$ of controls for $\mathcal{C}$ is a $(n\tau, Q, \text{int}Q)$-spanning set and hence

$$a_{n\tau}(f,Q) = \inf_{\mathcal{S}} \sum_{\omega\in\mathcal{S}} e^{(S_{n\tau}f)(\omega)} \leq \sum_{\omega\in\mathcal{W}_n} e^{(S_{n\tau}f)(\omega)},$$

where the infimum is taken over all $(n\tau, Q, \text{int}Q)$-spanning set $\mathcal{S}$. It follows that $a_{n\tau}(f,Q) \leq q_n(f,Q,\mathcal{C})$ and therefore

$$P_{\text{inv,int}}(f,Q) = \lim_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f,Q) \leq \lim_{n\to\infty} \frac{1}{n\tau} \log q_n(f,Q,\mathcal{C}) = P_{\text{fb}}(f,Q,\mathcal{C}).$$

Since this holds for every invariant open cover $\mathcal{C}$, we conclude

$$P_{\text{inv,int}}(f,Q) \leq \inf_{\mathcal{C}} P_{\text{fb}}(f,Q,\mathcal{C}) = P_{\text{fb}}(f,Q),$$

where the infimum is taken over all invariant open covers $\mathcal{C}$ of Q.

To show that $P_{\text{fb}}(f,Q) \leq P_{\text{inv,int}}(f,Q)$ we construct an invariant open cover for $\tau \in \mathbb{N}$. Let $\mathcal{S}$ be a $(\tau, Q, \text{int}Q)$-spanning set. For each $\omega \in \mathcal{S}$ consider

$$A(\omega) := \{x \in Q;\ \varphi(j,x,\omega) \in \text{int}Q \text{ for } j = 1,\ldots,\tau\}.$$

The set $\mathcal{A} = \{A(\omega);\ \omega \in \mathcal{S}\}$ forms a finite open cover of Q. Now define a map $G : \mathcal{A} \to U^\tau$ by

$$G(A(\omega)) = (\omega_0,\ldots,\omega_{\tau-1}).$$

Clearly, $\mathcal{C} := (\mathcal{A},\tau,G)$ is an invariant open cover of Q.

Recall that $\alpha \in \mathcal{A}^{\mathbb{N}_0}$ defines a control $\omega(\alpha)$ and for $n \in \mathbb{N}$ the set $B_n(\alpha)$ is given by

$$B_n(\alpha) := \{x \in X;\ \varphi(i\tau, x, \omega(\alpha)) \in A_i \text{ for } i = 0, 1, \ldots, n-1\}.$$

These sets form on open cover $\mathcal{B}_n = \mathcal{B}_n(\mathcal{C})$ of Q. Consider a generating set of feedback controls of the form

$$\mathcal{W}_n = \{\omega(\alpha_i); \alpha_i \in \mathcal{A}^{\mathbb{N}_0} \text{ for } i \in I\},$$

hence the sets $B_n(\alpha_i), i \in I$, form a subcover of $\mathcal{B}_n(\mathcal{C})$ which is minimal. Therefore

$$\begin{aligned}
\sum_{\omega \in \mathcal{W}_n} e^{(S_{n\tau} f)(\omega)} &= \sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\omega)} e^{(S_\tau f)(\theta^\tau \omega)} \cdots e^{(S_\tau f)(\theta^{(n-1)\tau} \omega)} \\
&\leq \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\omega)} \right) \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\theta^\tau \omega)} \right) \cdots \left(\sum_{\omega \in \mathcal{B}_n} e^{(S_\tau f)(\theta^{(n-1)\tau} \omega)} \right) \\
&\leq \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right)^n .
\end{aligned}$$

Since the previous inequality holds for all finite $(\tau, Q, \mathrm{int}Q)$-spanning sets $\mathcal{S}$, it follows that $q_n(f, Q, \mathcal{C}) \leq [a_\tau(f, Q)]^n$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
P_{\text{fb}}(f, Q, \mathcal{C}) &= \lim_{n \to \infty} \frac{1}{n\tau} \log q_n(f, Q, \mathcal{C}) \leq \lim_{n \to \infty} \frac{1}{n\tau} \log \left[a_\tau(f, Q) \right]^n \\
&= \frac{1}{\tau} \log a_\tau(f, Q).
\end{aligned}$$

Using Proposition 4.1.3 we conclude that

$$P_{\text{fb}}(f, Q) = \inf_{\mathcal{C}} P_{\text{fb}}(f, Q, \mathcal{C}) \leq \inf_{\tau \in \mathbb{N}} \frac{1}{\tau} \log a_\tau(f, Q) = P_{\text{inv,int}}(f, Q).$$

□

4.3 A Note on Transmission Data Rate

It is well known that the topological feedback entropy characterizes the smallest possible data rate that permits a specified compact set to be made invariant, by a causal coding and control law belonging to a general class (cf. [27, Theorem 2.1] and [32, Theorem 1]). Our goal in this section is to get a weighted version of this result by extending the definition of transmission data rate of a channel presented in [32] and relate this to the topological feedback pressure.

Suppose that a sensor, which is connected to a controller by a noiseless digital channel, measures the state at discrete sampling times τ_k, $k \geq 0$, say $\tau_k = k$. At time τ_k, one discrete-valued symbol s_k from a finite coding alphabet S_k of time-varying size is transmitted.

Each symbol transmitted by the coder may depend on all past and present states and past symbols, that is, we have a **coder mapping** $\gamma_k : X^{k+1} \times S_0 \times \cdots \times S_{k-1} \to S_k$, $\gamma_k(x_0, \cdots, x_k, s_0, \cdots, s_{k-1}) = s_k$. Assuming that the digital channel is errorless, at time τ_k, the controller has $s_0, \cdots, s_k$ available and generates a control value $u_k = \delta_k(s_0, \cdots, s_k)$, where δ_k is the **controller mapping** $\delta_k : S_0 \times \cdots \times S_k \to U$ We define the coder-controller as the triple

$$\mathcal{H} := (S, \gamma, \delta) = (\{S_k\}_{k\in\mathbb{N}_0}, \{\gamma_k\}_{k\in\mathbb{N}_0}, \{\delta_k\}_{k\in\mathbb{N}_0}).$$

The **generalized transmission data rate** of the channel with weight f is defined as the asymptotic weighted average bit rate

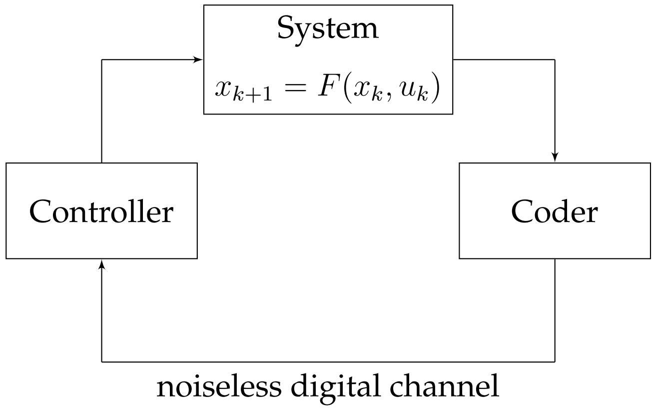
$$R(f, \mathcal{H}) := \liminf_{k\to\infty} \frac{1}{k} \log \left(\sum_{(s_0,\cdots,s_{k-1})\in\prod_{i=0}^{k-1} S_i} e^{(S_k^{\mathcal{H}} f)(s_0,\cdots,s_{k-1})} \right),$$

where

$$\begin{aligned}(S_k^{\mathcal{H}} f)(s_0, \cdots, s_{k-1}) &:= f(\delta_0(s_0)) + f(\delta_1(s_0, s_1)) + \cdots + f(\delta_{k-1}(s_0, \cdots, s_{k-1})) \\ &= \sum_{i=0}^{k-1} f(\delta_i(s_0, \cdots, s_i)).\end{aligned}$$

Remark 4.3.1. *Note that if we do not put any weight on the control values, i.e., if f is the null function $\boldsymbol{0}$, then the generalized transmission data rate $R(\boldsymbol{0}, \mathcal{H})$ coincide with the transmission data rate R presented in [32, Section III].*

Now, let $Q \subset X$ be a strongly invariant set. We say that a coder-controller (S, γ, δ) renders Q (strongly) invariant if for every $x_0 \in Q$, the sequence of states $(x_k)_{k\in\mathbb{N}_0}$ generated by the coder-controller satisfies $x_k \in \mathrm{int}Q$ for all $k \geq 1$.



Theorem 4.3.2. *For every $f \in C(U, \mathbb{R})$ we get*

$$P_{\mathrm{fb}}(f, Q) = \inf_{\mathcal{H}} R(f, \mathcal{H}),$$

where the infimum is taken over all coder-controllers $\mathcal{H} = (S, \gamma, \delta)$ that render Q strongly invariant.

Proof. Given $\varepsilon > 0$, there exists an invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$ such that $P_{\mathrm{fb}}(f, Q, \mathcal{C}) - P_{\mathrm{fb}}(f, Q) \leq \frac{\varepsilon}{2}$. Since

$$P_{\mathrm{fb}}(f, Q, \mathcal{C}) = \lim_{j \to \infty} \frac{1}{j\tau} \log q_j(f, Q, \mathcal{C}),$$

there is $j := j(\varepsilon) \in \mathbb{N}$ with

$$\frac{1}{j\tau} \log q_j(f, Q, \mathcal{C}) \leq P_{\mathrm{fb}}(f, Q, \mathcal{C}) + \frac{\varepsilon}{4}.$$

Fixing such j, let $\mathcal{W}_j = \{\omega(\alpha_1), \cdots, \omega(\alpha_m)\}$ be a generating set of feedback controls (of length $j\tau$) for $\mathcal{C}$ such that

$$\frac{1}{j\tau} \log \sum_{\omega \in \mathcal{W}_j} e^{(S_{j\tau} f)(\omega)} \leq \frac{1}{j\tau} \log q_j(f, Q, \mathcal{C}) + \frac{\varepsilon}{4},$$

(here we used that $\log$ is continuous and increasing), hence we obtain an open subcover $\{B(\alpha_1), \cdots, B(\alpha_m)\}$ of $\mathcal{B}_j$. We construct a periodic coding law using these possibly overlapping sets via the rule

$$s_k = \begin{cases} \min\{\sigma;\ x_k \in B(\alpha_\sigma)\}, & \text{if } \ k \in (j\tau)\mathbb{N}_0 \\ 1, & \text{otherwise} \end{cases}$$

In order to build the controller, note that once received the symbol $s_{l(j\tau)} = \sigma$ which index an open set $B(\alpha_\sigma)$, hence there are $A_0^\sigma, \ldots, A_{j-1}^\sigma \in \mathcal{A}$ such that

$$B(\alpha_\sigma) = \{x \in X;\ \varphi(r\tau, x, \omega(\alpha_\sigma)) \in A_r^\sigma,\ \text{for } r = 0, 1, \cdots, j-1\}.$$

Given $\sigma \in \{1, \cdots, m\}$, the controller will generate inputs via the periodic rule

$$\begin{aligned}\omega(\alpha'_\sigma) \quad = \quad & (\underbrace{u_0, \ldots, u_{\tau-1}}_{G(A_0^\sigma)}, \underbrace{u_\tau, \ldots, u_{2\tau-1}}_{G(A_1^\sigma)}, \ldots, \underbrace{u_{(j-1)\tau}, \ldots, u_{j\tau-1}}_{G(A_{j-1}^\sigma)}, \ldots \\ \cdots \quad & , \underbrace{u_{lj\tau}, \ldots, u_{(lj+1)\tau-1}}_{G(A_0^\sigma)}, \underbrace{u_{(lj+1)\tau}, \ldots, u_{(lj+2)\tau-1}}_{G(A_1^\sigma)}, \ldots, \underbrace{u_{((l+1)j-1)\tau}, \ldots, u_{(l+1)j\tau-1}}_{G(A_{j-1}^\sigma)}, \ldots)\end{aligned}$$

where α'_σ is the j-periodic sequence in $\mathcal{A}$ given by

$$\alpha'_\sigma := (A_0^\sigma, \cdots, A_{j-1}^\sigma, A_0^\sigma, \cdots, A_{j-1}^\sigma, \cdots).$$

It is important to note that for $i = 0, \ldots, j\tau - 1$, we have $\omega(\alpha_\sigma)_i = \omega(\alpha'_\sigma)_i = \delta_i(s_0, 1, \ldots, 1)$. Hence

$$(S_{n\tau} f)(\omega(\alpha_\sigma)) = (S_{n\tau} f)(\omega(\alpha'_\sigma)) = (S_{n\tau}^{\mathcal{H}} f)(s_0, 1, \ldots, 1).$$

By definition of invariant open covers this yields $x_{(lj+q)\tau} \in \text{int}Q$ and hence, the constructed coder-controller renders Q invariant.

Denoting by $\widetilde{\mathcal{W}}_j$ the set $\{\omega(\alpha'_1), \ldots, \omega(\alpha'_m)\}$, we obtain

$$\begin{aligned}
R(f,\mathcal{H}) &= \liminf_{k\to\infty} \frac{1}{k} \log \left(\sum_{(s_0,\cdots,s_{k-1})\in\prod_{i=0}^{k-1} S_i} e^{(S_k^{\mathcal{H}} f)(s_0,\cdots,s_{k-1})} \right) \\
&= \liminf_{k\to\infty} \frac{1}{k} \log \left(\sum_{(s_0,1,\cdots,1)\in\prod_{i=0}^{j\tau-1} S_i} e^{\lfloor k/(j\tau)\rfloor \left(S_{j\tau}^{\mathcal{H}} f\right)(s_0,1,\cdots,1)} \right) \\
&= \liminf_{k\to\infty} \frac{1}{k} \log \left(e^{\left\lfloor \frac{k}{j\tau} \right\rfloor} \sum_{\omega\in\widetilde{\mathcal{W}}_j} e^{(S_{j\tau} f)(\omega)} \right) \\
&= \liminf_{k\to\infty} \frac{1}{k} \left\lfloor \frac{k}{j\tau} \right\rfloor \log \left(\sum_{\omega\in\widetilde{\mathcal{W}}_j} e^{(S_{j\tau} f)(\omega)} \right) \\
&= \frac{1}{j\tau} \log \sum_{\omega\in\widetilde{\mathcal{W}}_j} e^{(S_{j\tau} f)(\omega)} = \frac{1}{j\tau} \log \sum_{\omega\in\mathcal{W}_j} e^{(S_{j\tau} f)(\omega)} \\
&\leq \frac{1}{j\tau} \log q_j(f,Q,\mathcal{C}) + \frac{\varepsilon}{4} \leq P_{\text{fb}}(f,Q,\mathcal{C}) + \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
R(f,\mathcal{H}) - P_{\text{fb}}(f,Q) &= (R(f,\mathcal{H}) - P_{\text{fb}}(f,Q,\mathcal{C})) + (P_{\text{fb}}(f,Q,\mathcal{C}) - P_{\text{fb}}(f,Q)) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

For the reverse inequality, let $\mathcal{H} := (S,\gamma,\delta)$ be a coder-controller that renders Q invariant. Given $\varepsilon > 0$, there exists $l \in \mathbb{N}$ such that

$$\frac{1}{l} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0,\cdots,s_{l-1})} \right) < R(f,\mathcal{H}) + \varepsilon. \tag{4.3-7}$$

Then, we can build a periodic coder-controller $\mathcal{H}^P := (S^P, \gamma^P, \delta^P)$ with period l as

$$\begin{aligned}
& S_k^P := S_{k \mod l} \\
& s_k = \gamma_k^P(\{x_i\}_{i=0}^k, \{s_i\}_{i=0}^{k-1}) := \gamma_{k \mod l}(\{x_i\}_{i=l\lfloor k/l\rfloor}^k, \{s_i\}_{i=l\lfloor k/l\rfloor}^{k-1}) \\
& u_k = \delta_k^P(\{s_i\}_{i=0}^k) := \delta_{k \mod l}(\{s_i\}_{i=l\lfloor k/l\rfloor}^k)
\end{aligned}$$

By construction, this new coder-controller also renders Q invariant. Writing each $k \in \mathbb{N}$ as $k = p_k l + q_k$ with $p_k \in \mathbb{N}_0$ and $q_k \in \{0,\cdots,l-1\}$, the associated generalized transmission data rate $R(f, \mathcal{H}^P)$ can be computed as

$$
\begin{aligned}
& R(f, \mathcal{H}^P) \\
= \quad & \liminf_{k\to\infty} \frac{1}{k} \log \left(\sum_{(s_0,\cdots,s_{k-1})\in\prod_{i=0}^{k-1} S_i^P} e^{(S_k^{\mathcal{H}} f)(s_0,\cdots,s_{k-1})} \right) \\
= \quad & \liminf_{k\to\infty} \frac{1}{k} \log \left(\sum_{(s_0,\cdots,s_{k-1})\in\prod_{i=0}^{k-1} S_{i \mod l}} e^{(S_k^{\mathcal{H}} f)(s_0,\cdots,s_{k-1})} \right) \\
\leq \quad & \liminf_{k\to\infty} \frac{1}{p_k l + q_k} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{p_k (S_l^{\mathcal{H}} f)(s_0,\cdots,s_{l-1}) + q_k \sup f} \right) \qquad (4.3\text{-}8) \\
= \quad & \liminf_{k\to\infty} \frac{q_k}{p_k l + q_k} \sup f + \liminf_{k\to\infty} \frac{p_k}{p_k l + q_k} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0,\cdots,s_{l-1})} \right) \\
= \quad & \frac{1}{l} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0,\cdots,s_{l-1})} \right)
\end{aligned}
$$

Analogously we can show that

$$
R(f, \mathcal{H}^P) \geq \frac{1}{l} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0,\cdots,s_{l-1})} \right),
$$

replacing the $\sup f$ by $\inf f$ in the inequality 4.3-8, hence we obtain the equality

$$
R(f, \mathcal{H}^P) = \frac{1}{l} \log \left(\sum_{(s_0,\cdots,s_{l-1})\in\prod_{i=0}^{l-1} S_i} e^{(S_l f)(s_0,\cdots,s_{l-1})} \right).
$$

With 4.3-7 this implies $R(f, \mathcal{H}^P) < R(f, \mathcal{H}) + \varepsilon$. Each sequence of symbols in $S_0 \times \cdots \times S_{l-1}$ defines a coding region in X which is defined as the set of all initial states x which force the coder to generate this sequence. The total number n of nonempty and disjoint coding regions is less than or equal to $\prod_{i=0}^{l-1} \#S_i$. Let $C_1, \cdots, C_n$ denote these coding regions and note that $Q \subset \cup_{i=1}^{n} C_i$. From $\mathcal{H}^P$ we can now construct an invariant open cover $\mathcal{C} = (\mathcal{A}, \tau, G)$ of Q as follows: The time τ is set to l. For every x_0 in one of the coding regions $C_i = C_i(c_0, \cdots, c_{l-1})$ there exists an open neighborhood $N(x_0)$ such that for every $y_0 \in N(x_0)$ the same sequence $(c_0, \cdots, c_{l-1})$ of symbols gives $y_1, \cdots, y_l \in \text{int}Q$, due to continuity of the transition map with respect to the state variable. Thus, we can "blow up" the sets C_i by setting $A_i = \cup_{x_0 \in C_i} N(x_0)$. This defines the open cover $\mathcal{A} = \{A_1, \cdots, A_n\}$ of Q. Finally, the mapping sequence G is defined by $G(A_i) :=$ the symbol sequence $(c_0, \cdots, c_{l-1})$ corresponding to the coding region C_i. By construction, it is clear that $(\mathcal{A}, \tau, G)$ is an invariant open cover. The pressure of $\mathcal{C} = (\mathcal{A}, \tau, G)$ can be estimated by

$$\begin{aligned} P_{\text{fb}}(f, Q, \mathcal{C}) &= \lim_{j \to \infty} \frac{1}{j\tau} \log q_j(f, Q, \mathcal{C}) \\ &\leq \lim_{j \to \infty} \frac{1}{j\tau} \log \left(\sum_{(s_0, \cdots, s_{l-1}) \in \prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0, \cdots, s_{l-1})} \right)^j \\ &= \frac{1}{l} \log \left(\sum_{(s_0, \cdots, s_{l-1}) \in \prod_{i=0}^{l-1} S_i} e^{(S_l^{\mathcal{H}} f)(s_0, \cdots, s_{l-1})} \right). \end{aligned}$$

Therefore

$$P_{\text{fb}}(f, Q) \leq P_{\text{fb}}(f, Q, \mathcal{C}) \leq R(f, \mathcal{H}^P) < R(f, \mathcal{H}) + \varepsilon.$$

Taking $\varepsilon \searrow 0$ we get $P_{\text{fb}}(f, Q) \leq R(f, \mathcal{H})$. $\square$

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